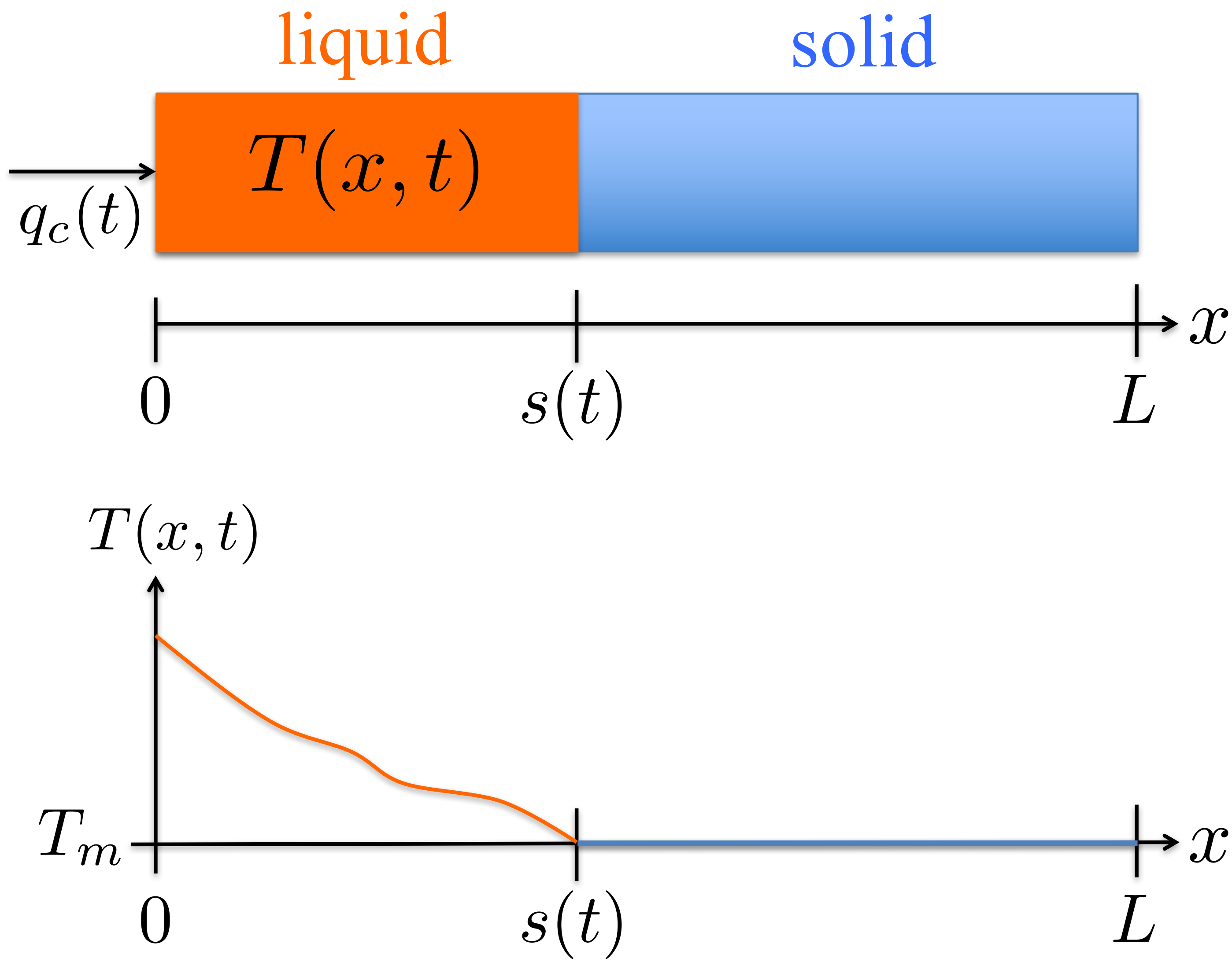
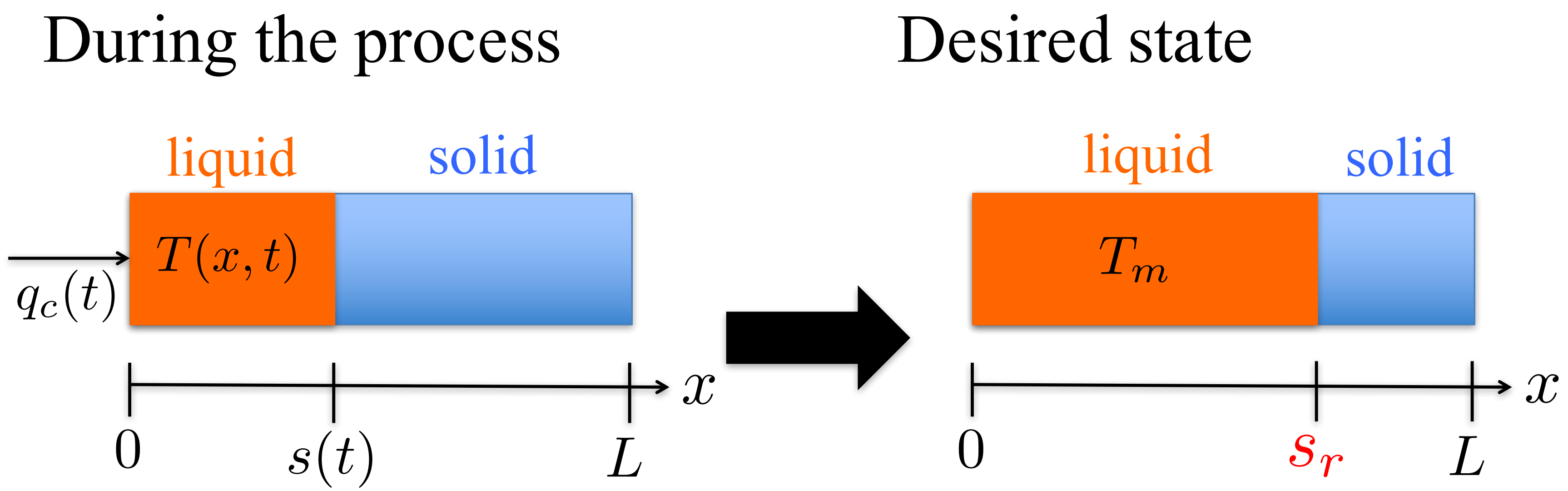
Backstepping Control of the One-Phase Stefan Problem

Shumon Koga, Mamadou Diagne, Shuxia Tang, Miroslav Krstic

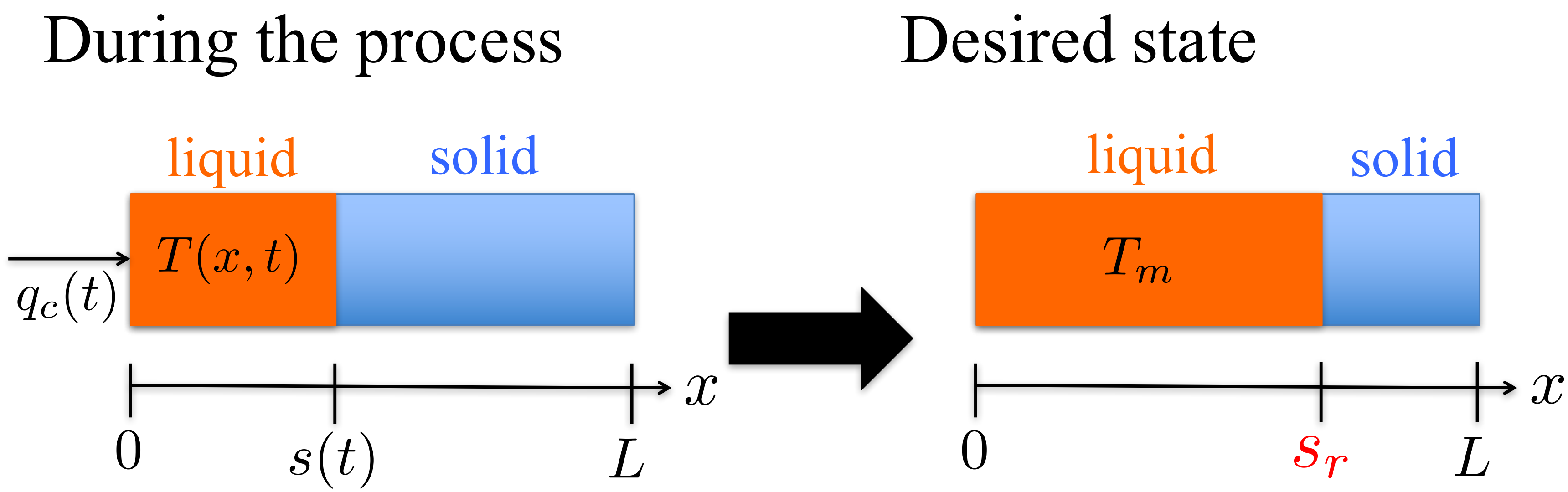
University of California, San Diego

ACC 2016


liquid
solid
$T(x,t)$
$q_c(t)$
0
$s(t)$
L
x
$T(x,t)$
T_m
0
$s(t)$
L
x

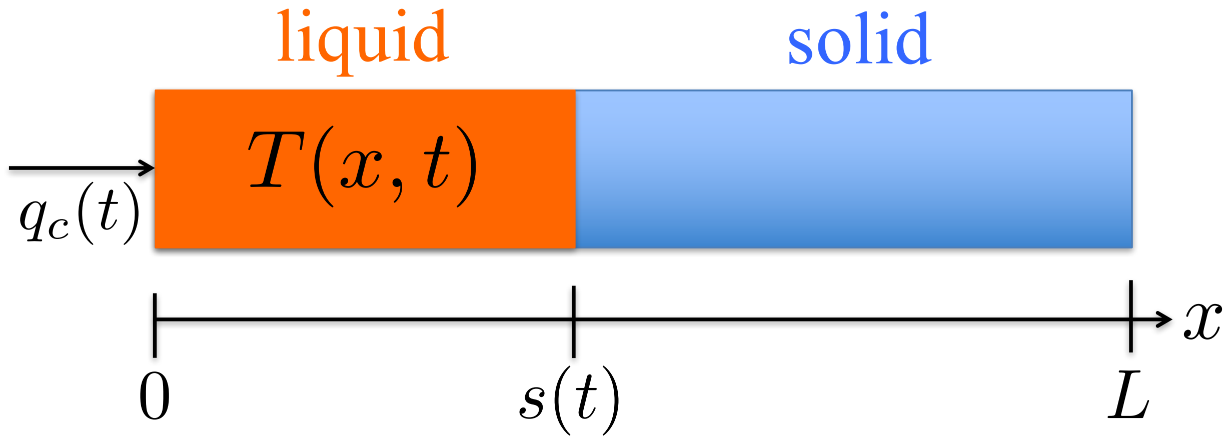


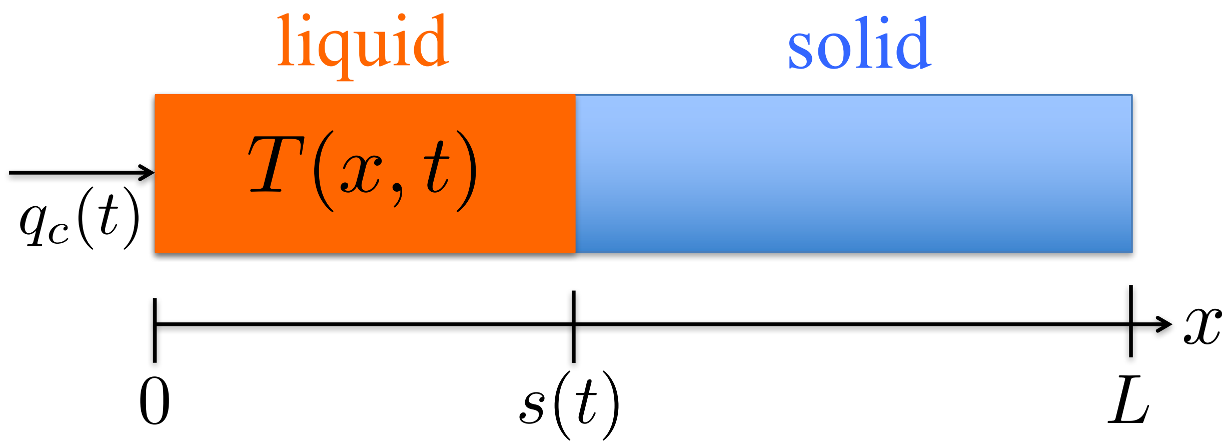
During the process
liquid
solid
$T(x,t)$
$q_c(t)$
0
$s(t)$
L
x
Desired state
liquid
solid
T_m
0
s_r
L
x



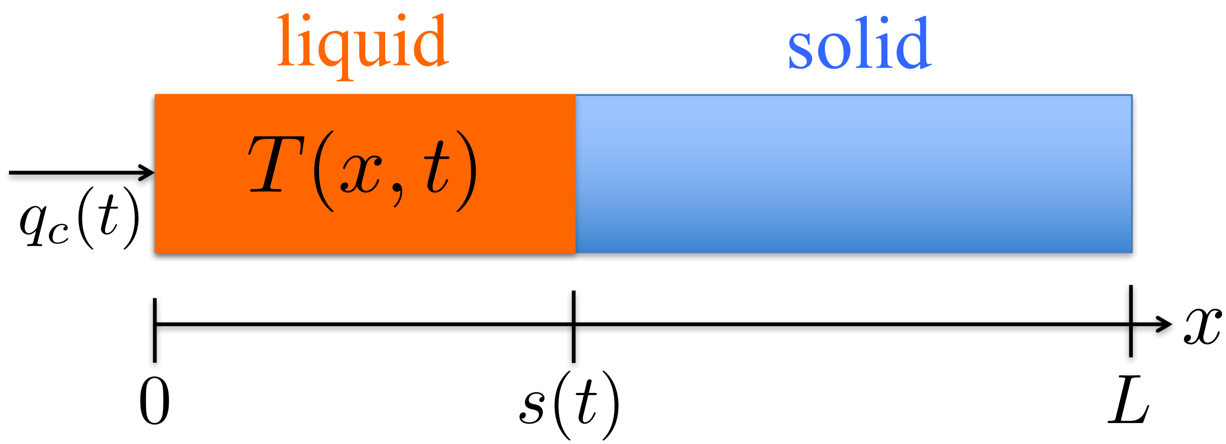
Objective: Design heat control $q_c(t) > 0$ to achieve

$$s(t) \to s_r, \quad T(x,t) \to T_m, \quad \text{as} \quad t \to \infty$$


liquid
solid
$T(x,t)$
$q_c(t)$
x
0
$s(t)$
L



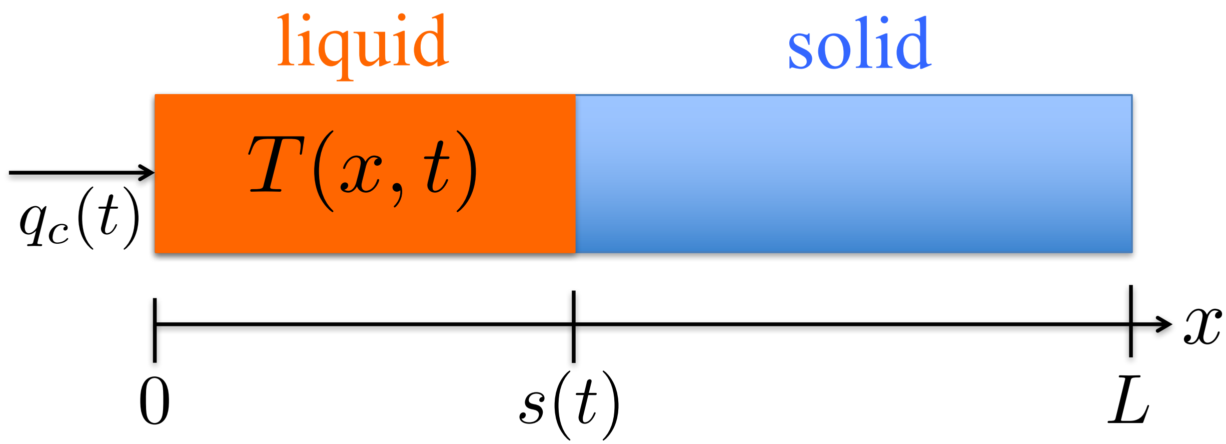
PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$



PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$

$$T_x(0,t) = -q_c(t)/k$$

$$T(s(t),t) = T_m$$

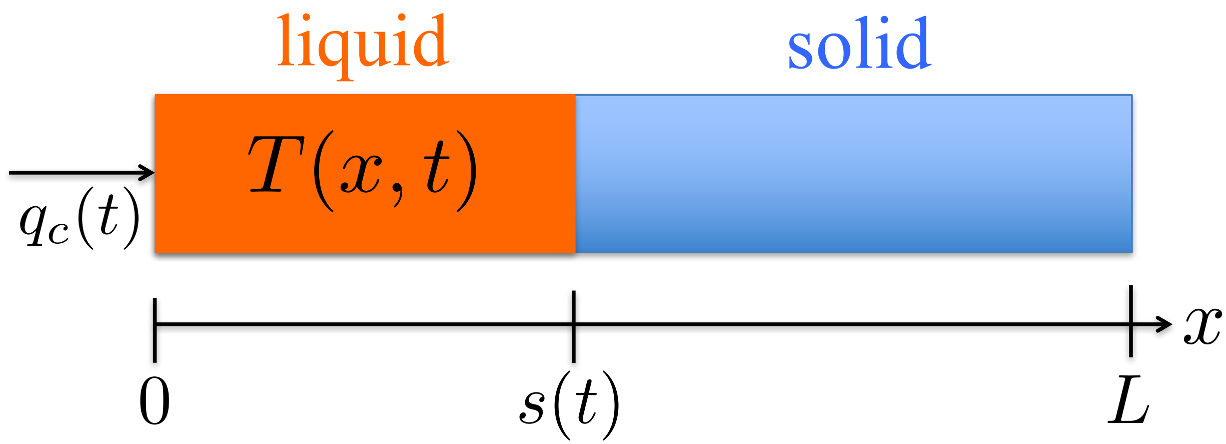


PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$

$$T_x(0,t) = -q_c(t)/k$$

$$T(s(t),t) = T_m$$

ODE $\quad \dot{s}(t) = -\beta T_x(s(t),t)$



PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$

$$T_x(0,t) = -q_c(t)/k$$

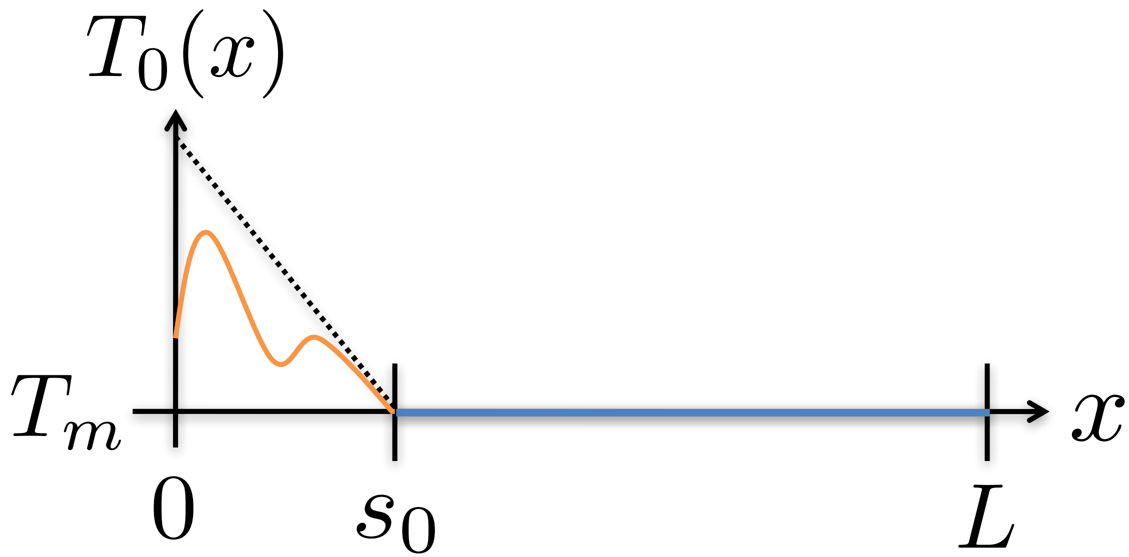
$$T(s(t),t) = T_m$$

ODE $\quad \dot{s}(t) = -\beta T_x(s(t),t)$

State-dependent moving boundary $\rightarrow$ Nonlinear

Assumption : Initial interface position $s_0 > 0$, and initial temperature $T_0(x)$ is Lipschitz (H := Lip. const.)

$$0 < T_0(x) - T_m < H(s_0 - x)$$



Model valid iff

$$T(x,t) > T_m, \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0$$

How to guarantee this?

Model valid iff

$$T(x,t) > T_m, \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0$$

How to guarantee this?

Lemma If $\boxed{q_c(t) > 0}$ $\forall t > 0$, then $\dot{s}(t) > 0$ $\forall t > 0$ and

$$T(x,t) > T_m, \ \forall x \in (0, s(t)), \forall t > 0$$

Energy conservation law

$$\frac{d}{dt}\left(\underbrace{\frac{1}{\alpha}\int_0^{s(t)}(T(x,t)-T_m)dx+\frac{1}{\beta}s(t)}_{\text{Internal Energy}}\right)=\underbrace{\frac{q_c(t)}{k}}_{\text{Work}}>0$$

Energy conservation law

$$\frac{d}{dt}\left(\underbrace{\frac{1}{\alpha}\int_0^{s(t)}(T(x,t)-T_m)dx+\frac{1}{\beta}s(t)}_{\text{Internal Energy}}\right)=\underbrace{\frac{q_c(t)}{k}}_{\text{Work}}>0$$

- For model to be valid (single melting interface), heat must be added.

Energy conservation law

$$\frac{d}{dt}\left(\underbrace{\frac{1}{\alpha}\int_0^{s(t)}(T(x,t)-T_m)dx+\frac{1}{\beta}s(t)}_{\text{Internal Energy}}\right)=\underbrace{\frac{q_c(t)}{k}}_{\text{Work}}>0$$

- For model to be valid (single melting interface), heat must be added.

- When heat added, internal energy grows.

Energy conservation law

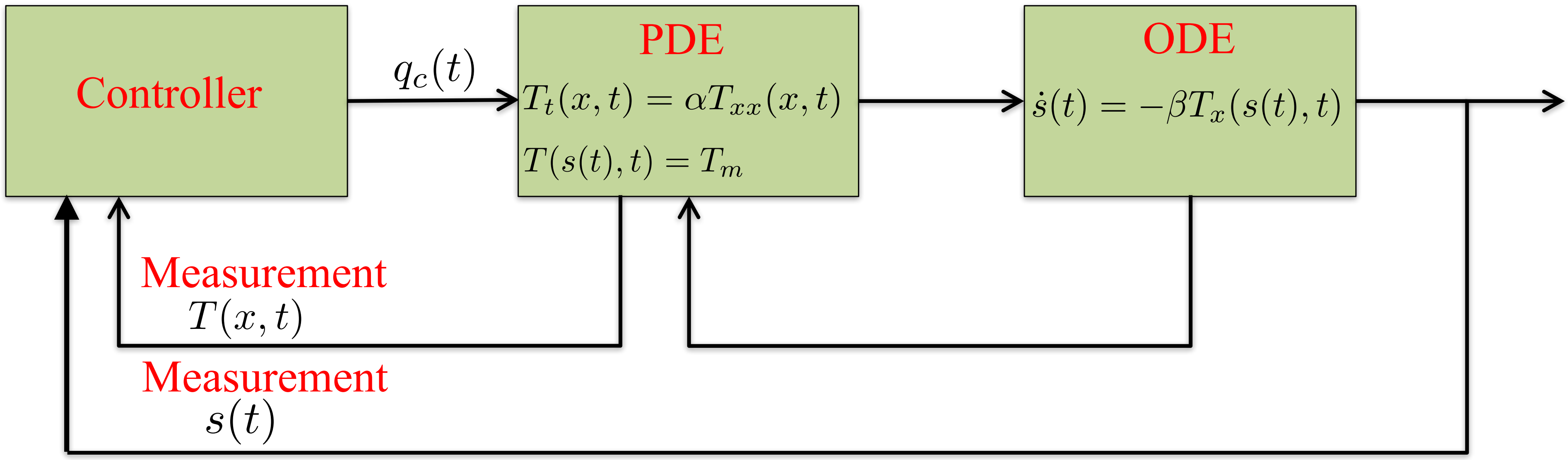
$$\frac{d}{dt}\left(\underbrace{\frac{1}{\alpha}\int_0^{s(t)}(T(x,t)-T_m)dx+\frac{1}{\beta}s(t)}_{\text{Internal Energy}}\right)=\underbrace{\frac{q_c(t)}{k}}_{\text{Work}}>0$$

- For model to be valid (single melting interface), heat must be added.
- When heat added, internal energy grows.
- Since internal energy grows, energy corresponding to setpoint must be greater than initial energy

The following assumption **necessary** $\left(\text{because } \int_0^{s_r} (T_r(x) - T_m)dx = 0\right)$

Assumption : Setpoint s_r chosen to satisfy

$$s_r > s_0 + \frac{\beta}{\alpha} \int_0^{s_0} (T_0(x) - T_m)dx \qquad =: \underline{s}_r(s_0, T_0) > s_0$$


Controller
$q_c(t)$
PDE
$T_t(x,t) = \alpha T_{xx}(x,t)$
$T(s(t),t) = T_m$
ODE
$\dot{s}(t) = -\beta T_x(s(t),t)$
Measurement
$T(x,t)$
Measurement
$s(t)$

Theorem The control law

$$q_c(t) = -ck\left(\frac{1}{\alpha}\int_0^{s(t)} (T(x,t) - T_m)dx + \frac{1}{\beta}(s(t) - s_r)\right)$$

where $c > 0$, makes the closed-loop system globally exponentially stable in the norm

$$||T - T_m||^2_{\mathcal{H}_1} + (s - s_r)^2.$$

Theorem The control law

$$q_c(t) = -ck\left(\frac{1}{\alpha}\int_0^{s(t)}(T(x,t)-T_m)dx + \frac{1}{\beta}(s(t)-s_r)\right)$$

where $c > 0$, makes the closed-loop system globally exponentially stable in the norm

$$||T - T_m||^2_{\mathcal{H}_1} + (s - s_r)^2.$$

Note : Control law is nonlinear because of $s(t)$ in integration limit.

Explanation of Design

Reference errors

$$u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r$$

Explanation of Design

Reference errors

$$u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r$$

Backstepping transformation

$$w(x,t) = u(x,t) - \frac{c}{\alpha}\int_x^{s(t)} (x-y)u(y,t)dy + \frac{c}{\beta}(s(t)-x)X(t)$$

Explanation of Design

Reference errors

$$u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r$$

Backstepping transformation

$$w(x,t) = u(x,t) - \frac{c}{\alpha}\int_x^{s(t)} (x-y)u(y,t)dy + \frac{c}{\beta}(s(t)-x)X(t)$$

Target system

$$\begin{aligned} w_t(x,t) =& \alpha w_{xx}(x,t) + \frac{c}{\beta}\dot{s}(t)X(t) \\ w(s(t),t) =& 0 \\ w_x(0,t) =& 0 \end{aligned}$$

$$\dot{X}(t) = -cX(t) - \beta w_x(s(t),t)$$

Model validity

Proposition Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$.

Model validity

Proposition Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$.

Proof:

$$q_c = -ck\,(\text{internal energy} - \text{setpoint internal energy})$$

$$\dot{q}_c(t) = -cq_c(t) \quad \therefore q_c(t) = q_c(0)e^{-ct} > 0$$

Model validity

Proposition Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$.

Proof:

$$q_c = -ck \text{ (internal energy} - \text{setpoint internal energy)}$$

$$\dot{q}_c(t) = -cq_c(t) \quad \therefore q_c(t) = q_c(0)e^{-ct} > 0$$

$q_c(t) > 0$ is a consequence (bonus, but not goal) of *backstepping* design!

Lyapunov analysis

$$V_1 := ||w||^2_{\mathcal{H}_1} + pX^2$$

$$\begin{aligned}\dot{V}_1 &\leq -bV_1 + \dot{s}(t)\left(m_1 X(t)||w||_{\mathcal{L}_2} - m_2 w_x(s(t),t)^2\right) \\ &\leq -bV_1 + a\dot{s}(t)V_1, \quad \because \dot{s}(t) > 0\end{aligned}$$

Overall Lyapunov functional

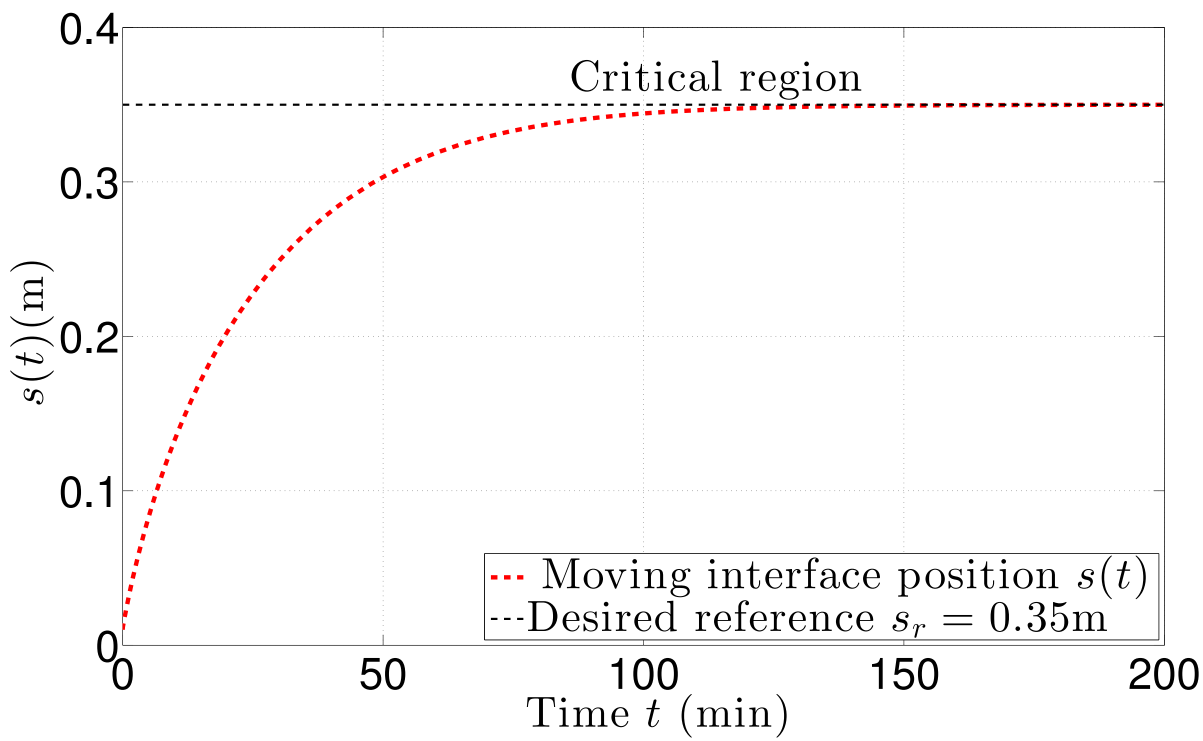
$$V := \frac{V_1}{e^{as}} = \frac{||w(T,s)||^2_{\mathcal{H}_1} + p(s-s_r)^2}{e^{as}}$$

yields $\dot{V} \leq -bV$, which leads to

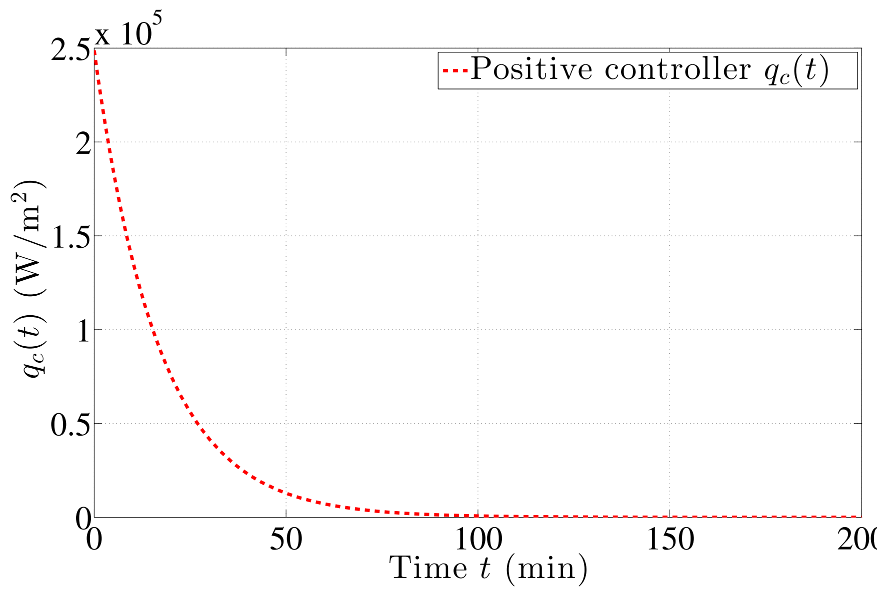
$$V_1 \leq e^{a(s(t)-s_0)} V_1(0) e^{-bt} \underbrace{\leq}_{\because\, s_0 < s(t) < s_r} e^{a(s_r - s_0)} V_1(0) e^{-bt}$$

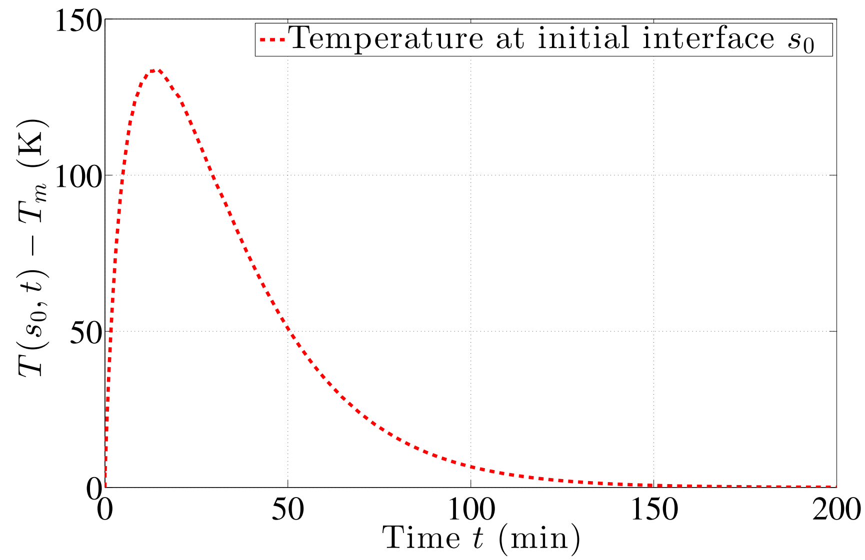
Numerical Simulation

Zinc

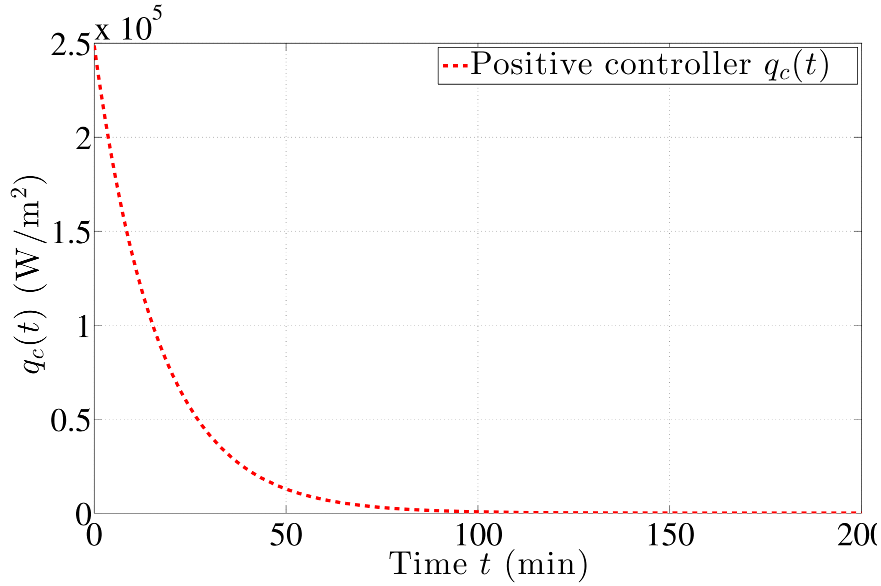


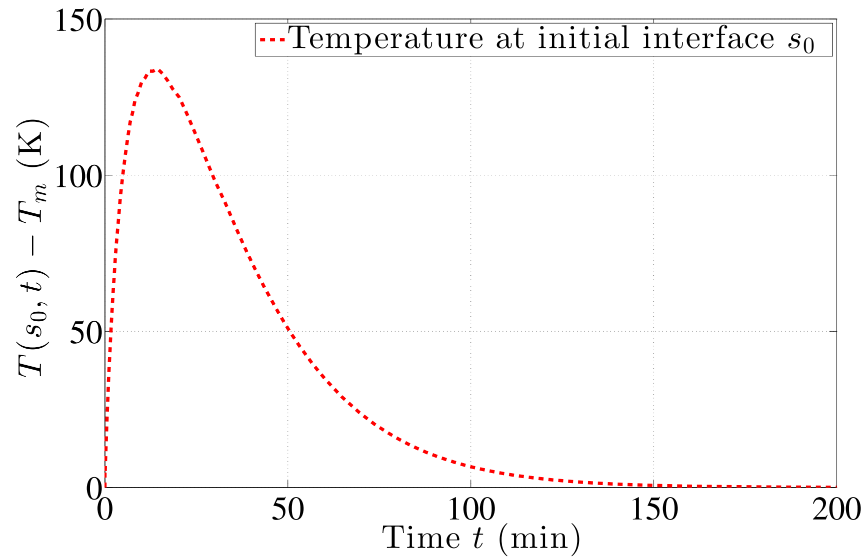
No overshoot





Control monotonic but temperature at initial interface not monotonic.





Control monotonic but temperature at initial interface not monotonic.

Warms up from freezing/melting temperature T_m, melts the ice to s_r, and returns to freezing/melting temperature T_m.