Backstepping Control of Stefan Problem with Flowing Liquid

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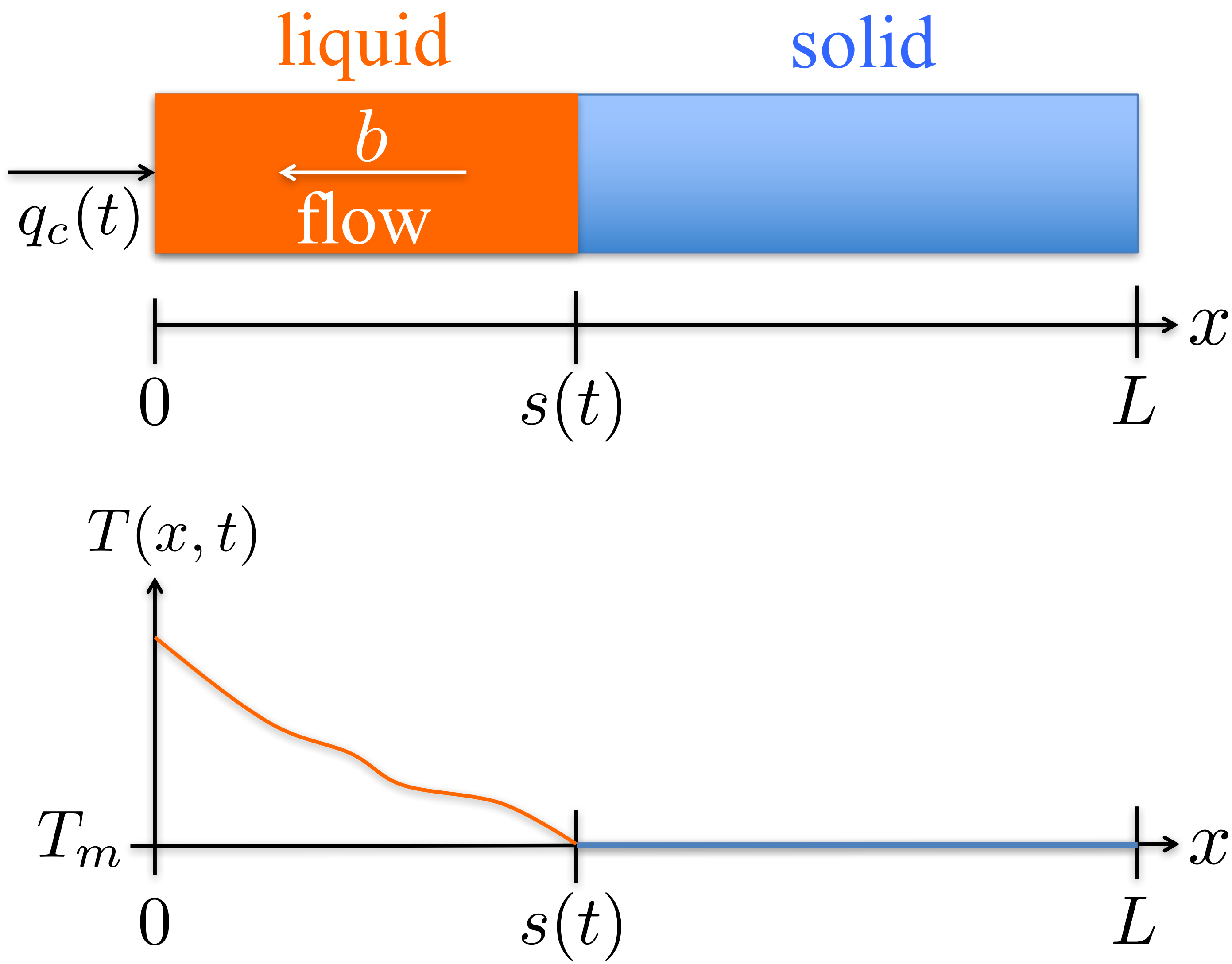
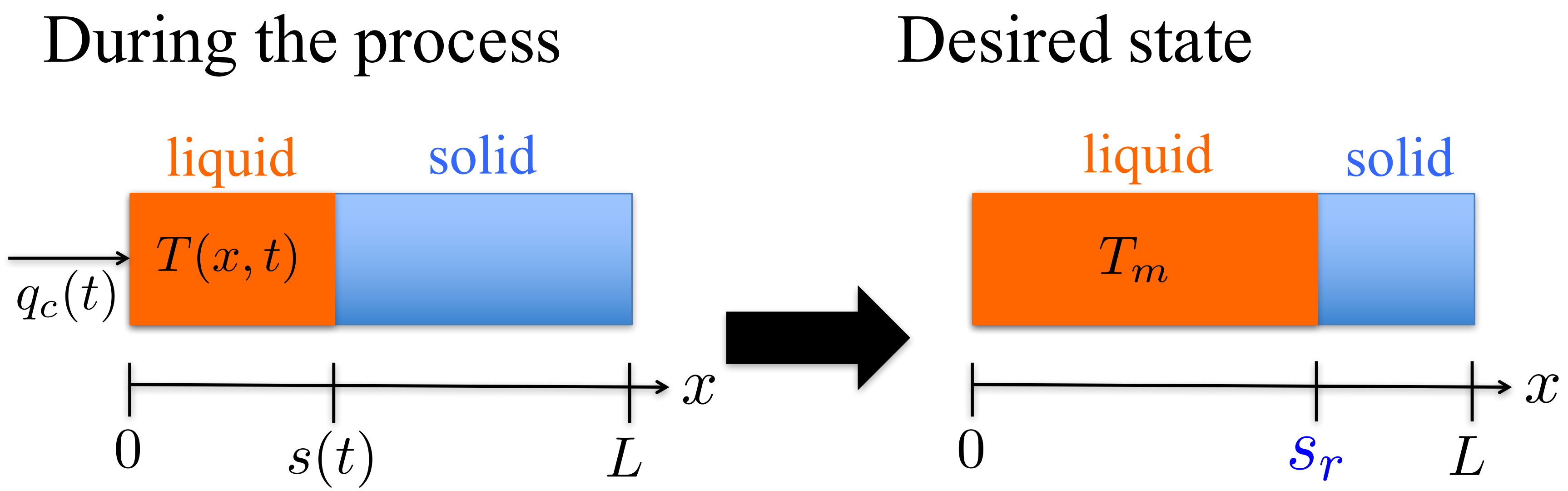
Outline

- Problem Statement
- Control Design
- Simulation & Future Work

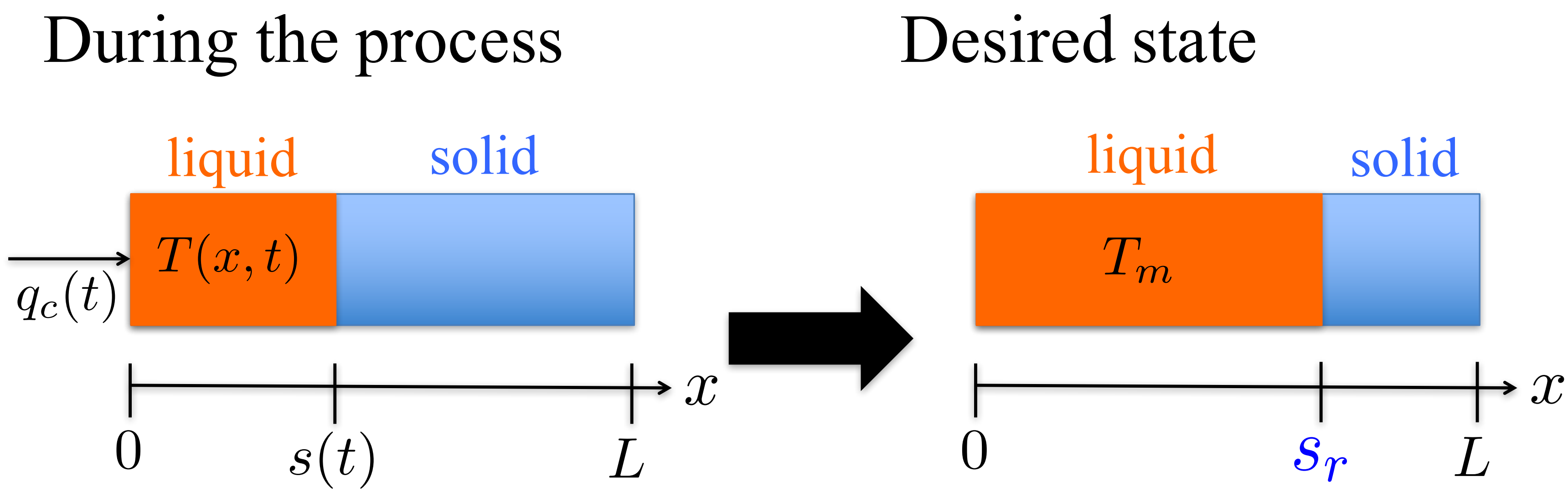
Problem Statement

Physical Model : Melting + Flow



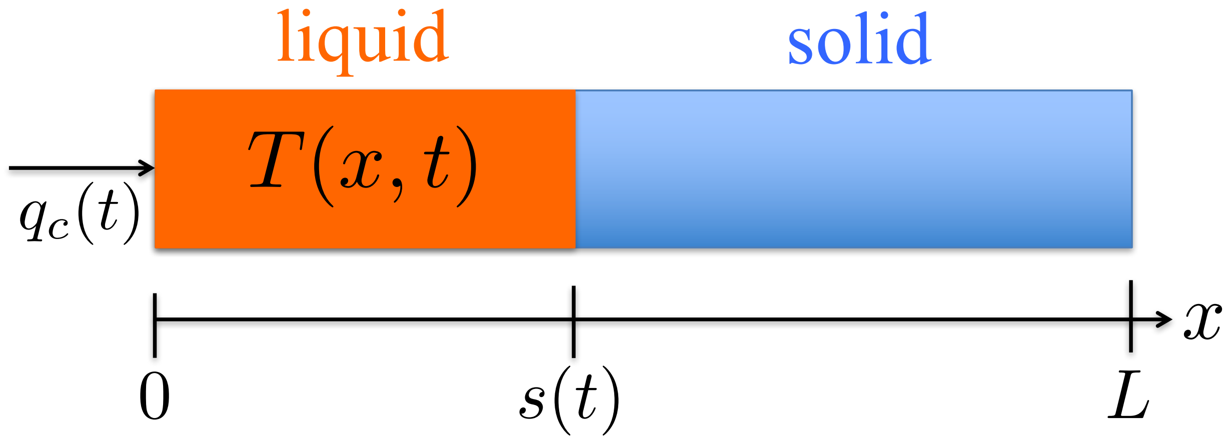


During the process
Desired state
liquid
solid
$T(x,t)$
$q_c(t)$
x
0
$s(t)$
L
liquid
solid
T_m
x
0
s_r
L



Objective: Design heat control $q_c(t)$ to achieve

$$s(t) \to s_r, \quad T(x,t) \to T_m, \quad \text{as} \quad t \to \infty$$



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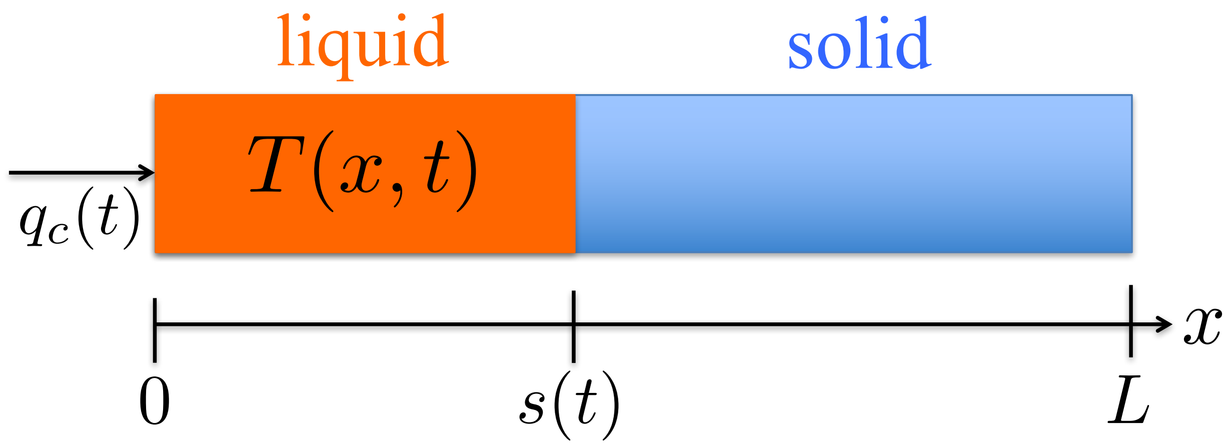
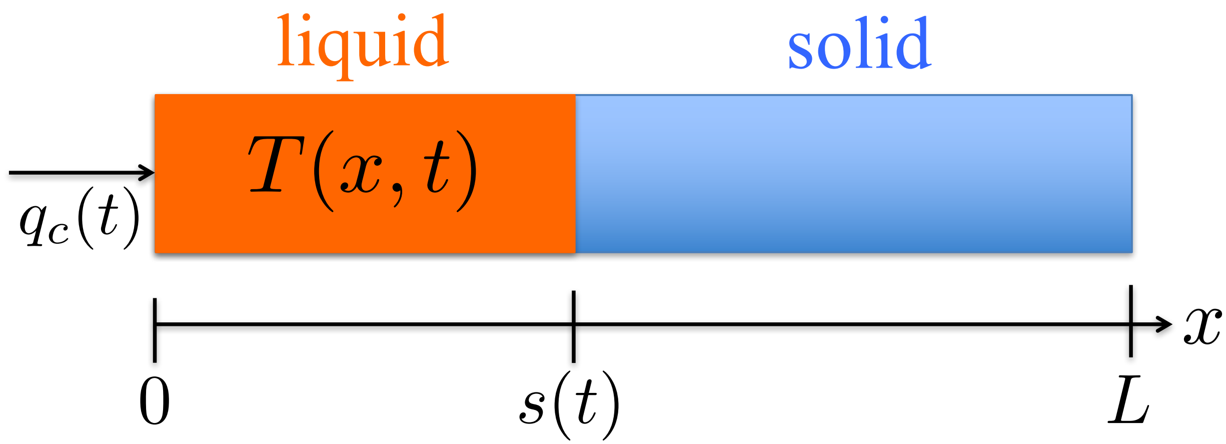


PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t) + bT_x(x,t), \quad 0 < x < s(t) < L$



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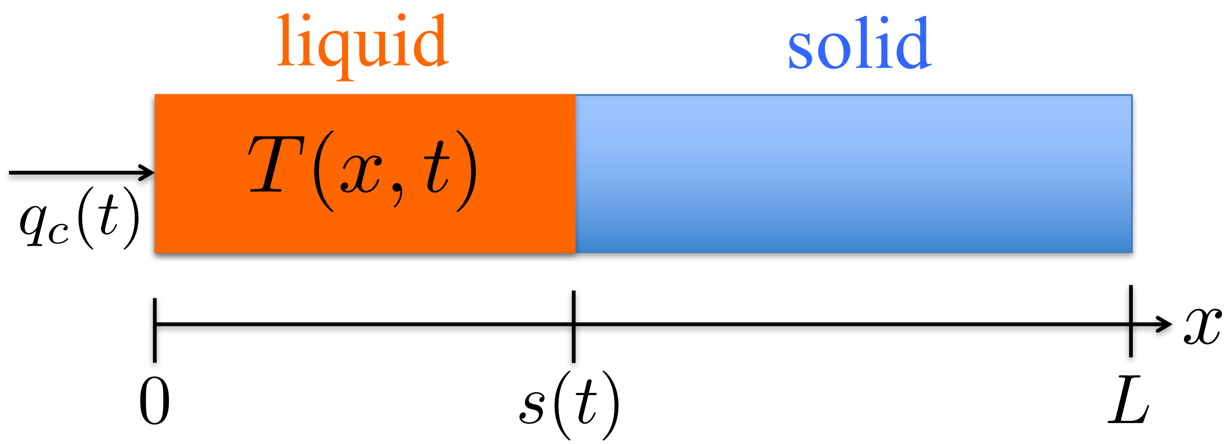
$$T(s(t),t) = T_m$$



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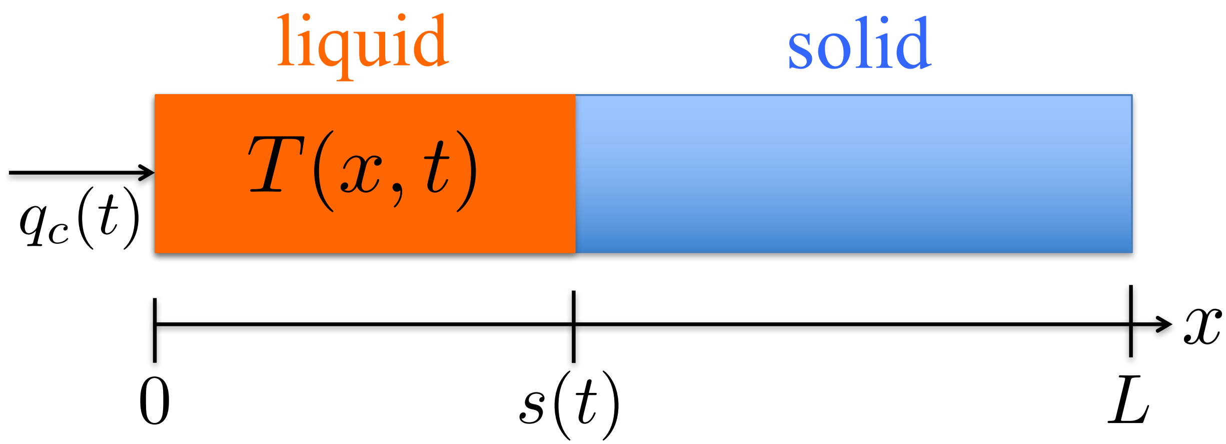
ODE $\quad \dot{s}(t) = -\beta T_x(s(t),t)$



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ODE
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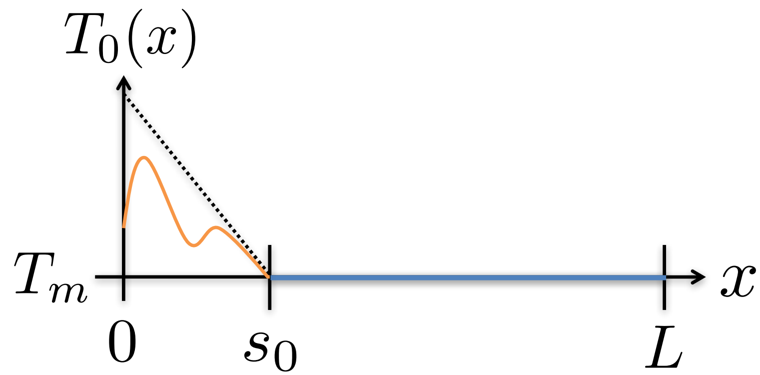
State-dependent moving boundary $\rightarrow$ Nonlinear

Assumption : Initial interface position $s_0 > 0$, and initial temperature $T_0(x)$ is Lipschitz (H := Lip. const.)

$$0 < T_0(x) - T_m < H(s_0 - x)$$



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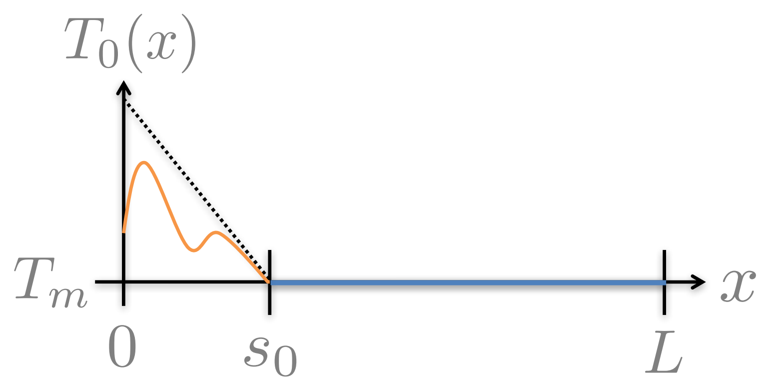

Remark : Model valid iff

$$T(x,t) > T_m, \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0$$

Energy Conservation

$$\frac{d}{dt}\left(\frac{1}{\alpha}\int_0^{s(t)} (T(x,t) - T_m)\,dx + \frac{1}{\beta}s(t)\right) = \frac{q_c(t)}{k} - \frac{b}{\alpha}\,(T(0,t) - T_m)$$

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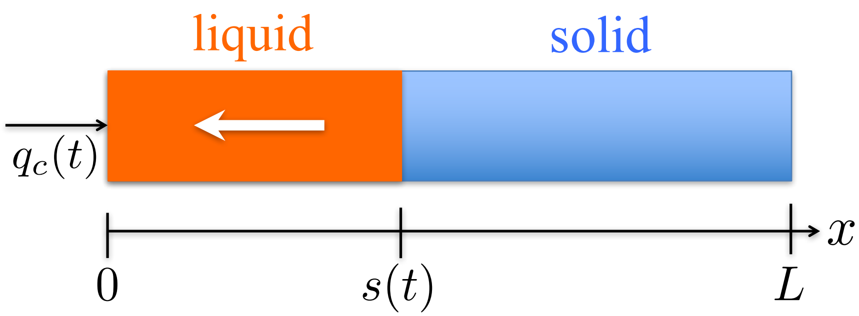

Neutralizes the energy growth

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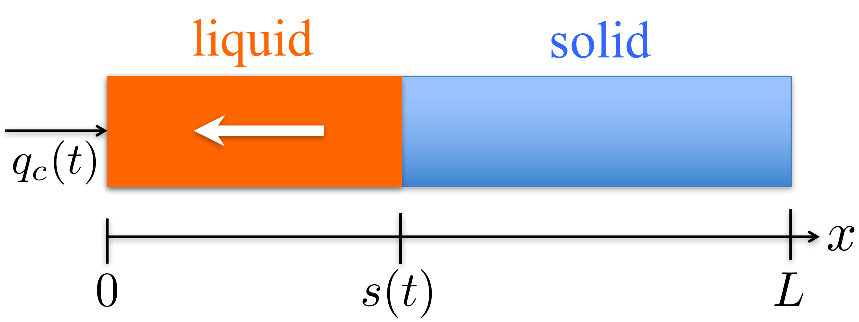

Neutralizes the energy growth

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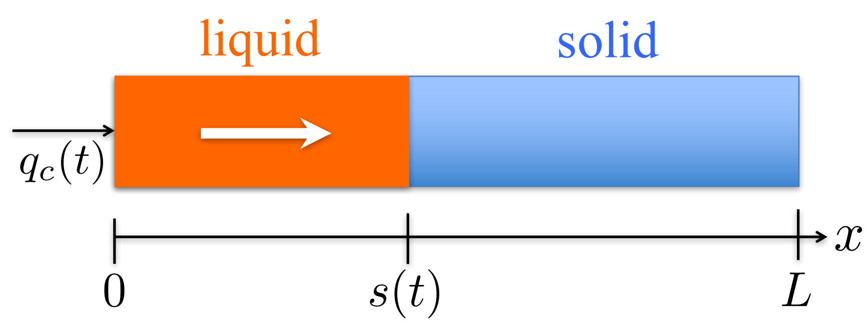

Promotes the energy growth

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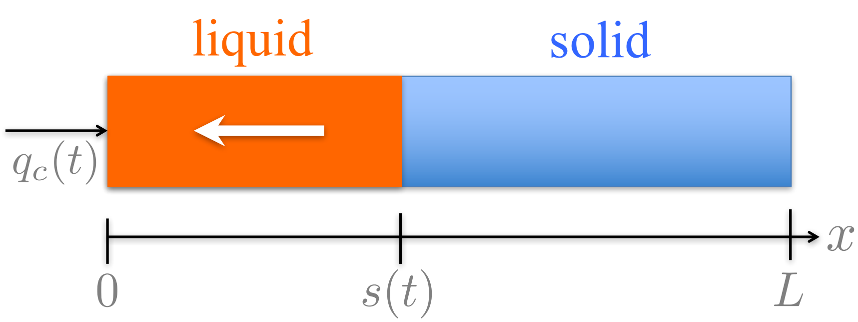

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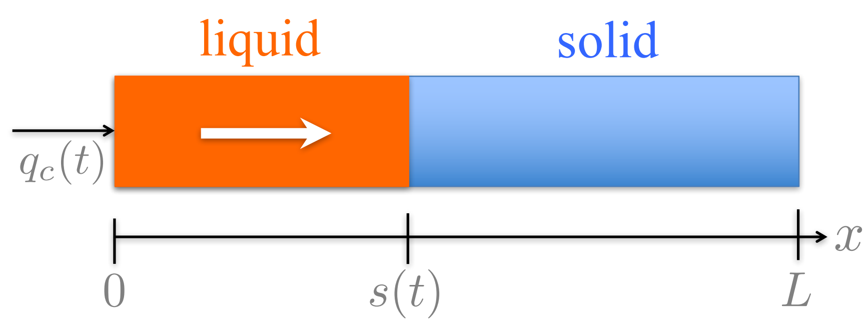

Promotes the energy growth

Intuition : s_r should be chosen sufficiently far from s_0 depending on $T_0(x)$ and b.

Main Results

Theorem 1 [Counter-convection]

Suppose $b > 0$. Consider the control law

$$q_c(t) = \frac{kb}{2\alpha}(T(0,t) - T_m) - ck\left(\frac{1}{\alpha}\int_0^{s(t)} \cosh\left(\frac{b}{2\alpha}x\right) e^{\frac{b}{2\alpha}x}(T(x,t) - T_m)dx \right.$$
$$\left. + \frac{2\alpha}{b\beta}\cosh\left(\frac{b}{2\alpha}s(t)\right)\left(e^{\frac{b}{2\alpha}s(t)} - e^{\frac{b}{2\alpha}s_r}\right)\right),$$

Then, for any s_r verifying setpoint restriction

$$s_r > s_0 + \frac{2\alpha}{b}\ln\left(1 + \frac{b\beta}{2\alpha^2}\int_0^{s_0}(T_0(x) - T_m)\,dx\right),$$

the setpoint (T_m, s_r) is exponentially stable in $\mathcal{H}_1$-norm.

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Note : As $b \to \infty$, the restriction is relaxed to $s_r > s_0$.

Theorem 2 [Regular-convection]

Suppose $b < 0$. Consider the same control law as counter-convection.

Assume $(T_0(x), s_0)$ satisfy initial condition requirement

$$\int_0^{s_0} (T_0(x) - T_m)\, dx < \frac{2\alpha^2}{\beta|b|} e^{-\frac{|b|}{2\alpha}s_0}.$$

Then, for any s_r verifying setpoint restriction

$$s_r > s_0 - \frac{2\alpha}{|b|} \ln\left(1 - \frac{|b|\beta}{2\alpha^2} e^{\frac{|b|}{2\alpha}s_0} \int_0^{s_0} (T_0(x) - T_m)\, dx\right),$$

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	$b > 0$	$b = 0$	$b < 0$
I.C. Requirement	None	None	Temp. cool
Setpoint Restriction	Less than $\star$	$\star$	More than $\star$

Control Design

Change of variables

Reference errors

$$u(x,t) = (T(x,t) - T_m)\, e^{\frac{b}{2\alpha}x}, \quad X(t) = \frac{2\alpha}{b}\left(e^{\frac{b}{2\alpha}s(t)} - e^{\frac{b}{2\alpha}s_r}\right).$$

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(u, X)-system

$$\text{PDE} \qquad u_t(x,t) = \alpha u_{xx}(x,t) - \lambda u(x,t), \quad 0 < x < s(t)$$

$$u_x(0,t) = -U(t), \quad u(s(t),t) = 0,$$

$$\text{ODE} \qquad \dot{X}(t) = -\beta u_x(s(t),t),$$

where

$$\lambda := \frac{b^2}{4\alpha}, \quad U(t) := \frac{q_c(t)}{k} - \frac{b}{2\alpha}(T(0,t) - T_m).$$

Model validity

Lemma If $\boxed{U(t) > 0}$ $\forall t > 0$, then $u(x,t) > 0$ & $\dot{s}(t) > 0$ for $\forall x \in (0, s(t))$ $\forall t > 0$ which verifies

$$T(x,t) > T_m, \ \forall x \in (0, s(t)), \forall t > 0$$

Proof is by maximum principle

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Proof is by maximum principle

Lemma With $b > 0$ (counter-convection), $U(t) > 0$ implies $\boxed{q_c(t) > 0}$.

Note : With $b < 0$ (regular-convection), $U(t) > 0$ *does not ensure* $q_c(t) > 0$.

Design Procedure

- Backstepping transformation

$$w(x,t) = u(x,t) - \frac{\beta}{\alpha}\int_x^{s(t)} \phi(x-y)u(y,t)dy - \phi(x-s(t))X(t),$$

where $\phi(x) = \frac{c}{\beta}\sqrt{\frac{\alpha}{\lambda}}\text{sinh}\left(\sqrt{\frac{\lambda}{\alpha}}x\right)$.

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- Target (w, X)-system

$$\begin{aligned} \text{PDE} \qquad w_t(x,t) =& \alpha w_{xx}(x,t) - \lambda w(x,t) + \dot{s}(t)\phi'(x-s(t))X(t), \\ w_x(0,t) =& 0, \quad w(s(t),t) = 0, \\ \text{ODE} \qquad \dot{X}(t) =& -cX(t) - \beta w_x(s(t),t), \end{aligned}$$

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$$w_x(0,t) = 0, \quad w(s(t),t) = 0,$$
$$\text{ODE} \quad \dot{X}(t) = -cX(t) - \beta w_x(s(t),t),$$

- Control design (*nonlinear*)

$$U(t) = -c\left(\frac{1}{\alpha}\int_0^{s(t)} \cosh\left(\sqrt{\frac{\lambda}{\alpha}}x\right)u(x,t)dx + \frac{1}{\beta}\cosh\left(\sqrt{\frac{\lambda}{\alpha}}s(t)\right)X(t)\right)$$

Proposition Provided $U(0) > 0$, the followings hold

$$U(t) > 0,$$ $*$ $q_c(t) > 0$ with $b > 0$

$$u(x,t) > 0, \quad \dot{s}(t) > 0,$$ $*$ model valid for both b

$$s_0 < s(t) < s_r,$$ $*$ no overshoot for both b

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Lemma Target (w, X)-sys. is exp. stable with $\dot{s}(t) > 0$ & $s_0 < s(t) < s_r$.
Proof : Lyapunov analysis

$$\Downarrow$$

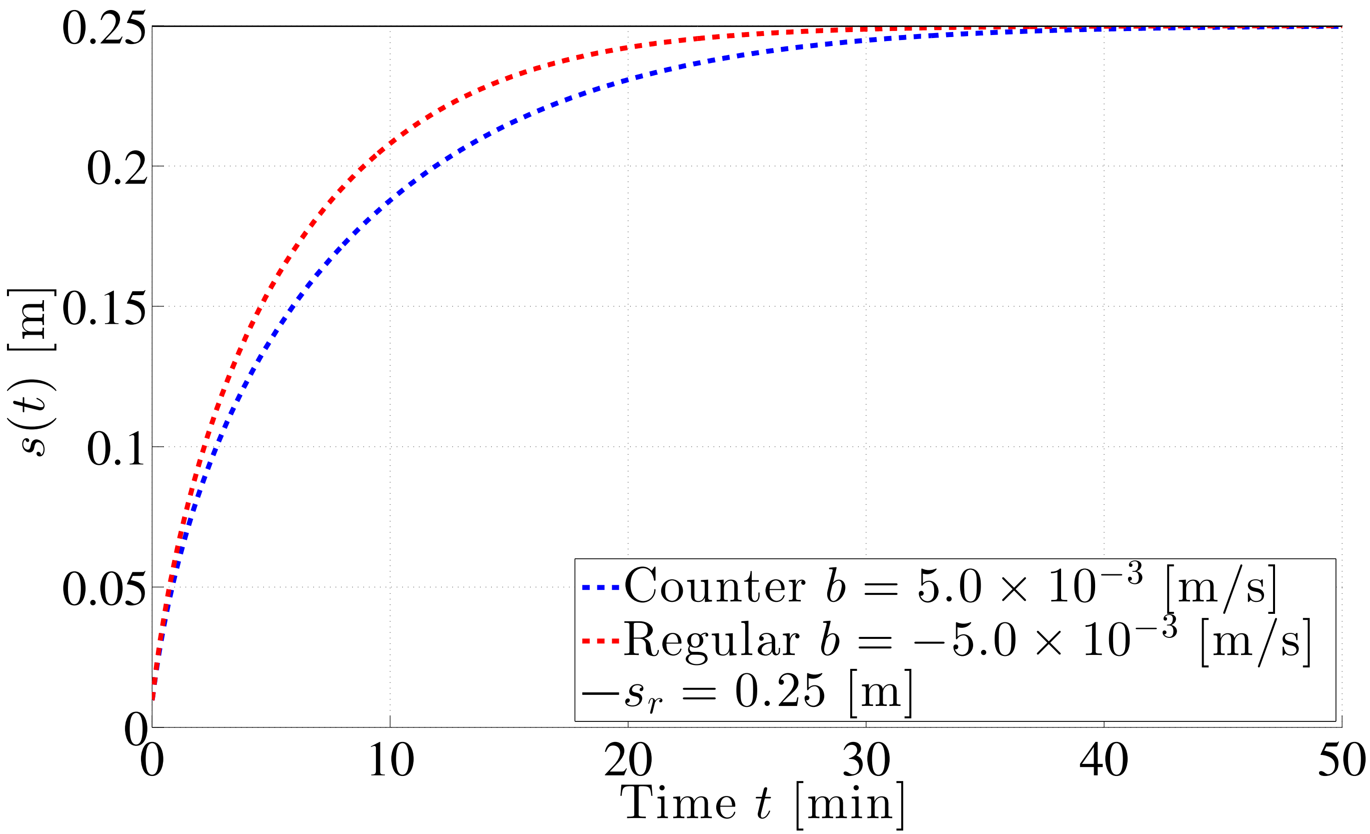
Lemma (T, s)-sys. is *exp. stable* at the setpoint (T_m, s_r).
Proof : Invertibility of transformations with $s_0 < s(t) < s_r$

Concludes the theorems.

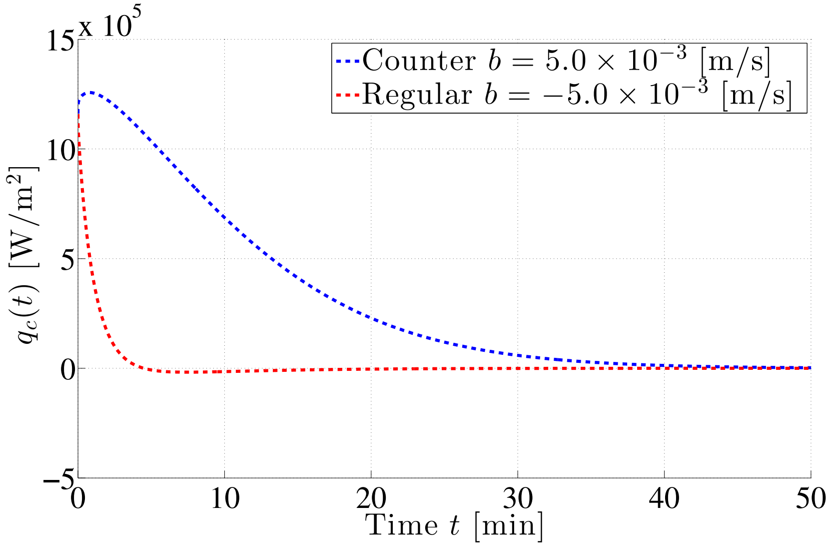
Simulation & Future Work

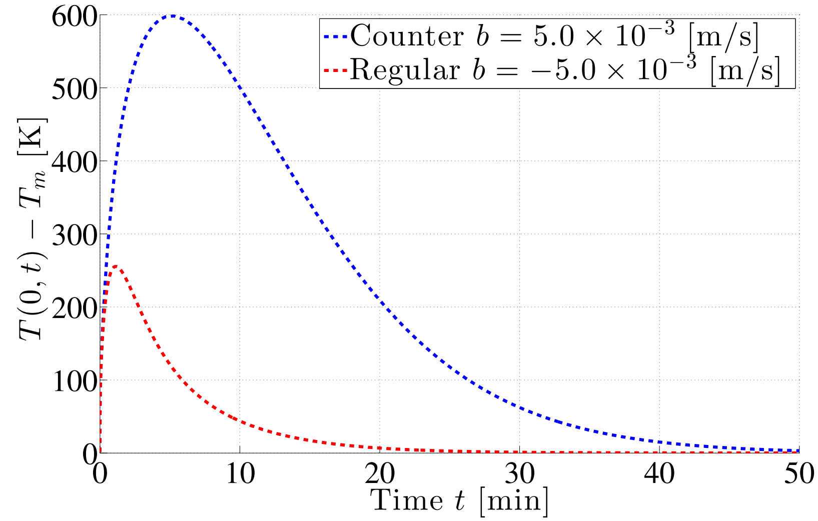
Numerical Simulation

Zinc



$*$ No overshoot



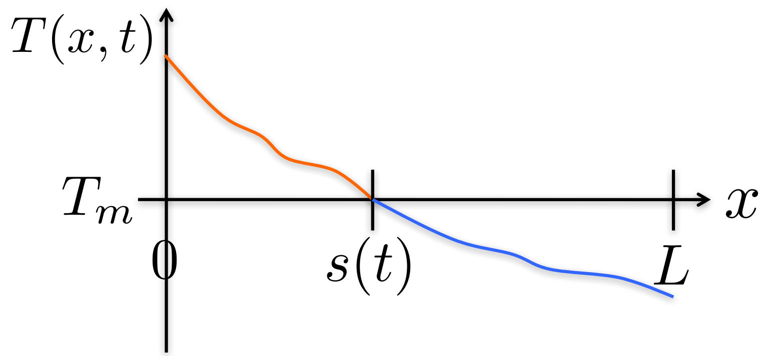


$*$ Positive heat with $b > 0$,
while with $b < 0$ it can be negative.

$*$ Temperature warms up
and cool into melting point.

Future Work

- Two-phase Stefan problem



- Extrusion for 3D-printing