Arctic Sea Ice Temperature Profile Estimation via Backstepping Observer Design

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What Is Sea Ice?

- Sea ice is **frozen ocean water**. (while icebergs, glaciers, etc. originate in land)

- It covers **12%** of the ocean.
Why Is Arctic Sea Ice Important?

• Affects global climate by reflection of solar energy.
Why Is Arctic Sea Ice Important?

- Affects global climate by reflecting the solar energy.

Ice-albedo positive feedback
Why Is Arctic Sea Ice Important?

- Recent decline of Arctic sea ice
Sea Ice in Global Climate Model

atmosphere

ocean <--> sea ice

Global climate model
Sea Ice in Global Climate Model

- Time-evolution of thickness and temperature by Maykut and Untersteiner, 1971 (MU71)
- Time-evolution of thickness distribution by Thorndike, et al, 1975
Sea Ice in Global Climate Model

- Time-evolution of thickness and temperature by Maykut and Untersteiner, 1971 (MU71)

- Time-evolution of thickness distribution by Thorndike, et al, 1975
Thermodynamic Model by MU71

\[(T_s(x, t), T_i(x, t)) \cdots \text{Temp. of snow, sea ice}\]

\[-h(t) \quad 0 \quad H(t)\]

\[x\]

air

snow

sea-ice

ocean
Thermodynamic Model by MU71

\[ F_a - \sigma(T_s(-h(t), t) + 273)^4 + k_s \frac{\partial T_s}{\partial x}(-h(t), t) = \begin{cases} 0, & \text{if } T_s(-h(t), t) < T_{m1}, \\ -q h(t), & \text{if } T_s(-h(t), t) = T_{m1}, \end{cases} \]

\((T_s(x, t), T_i(x, t)) \cdot \cdot \cdot \text{Temp. of snow, sea ice} \]

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\end{cases} \]
\[ \rho_s c_0 \frac{\partial T_s}{\partial t} = k_s \frac{\partial^2 T_s}{\partial x^2}(x, t), \quad -h(t) < x < 0, \]

\((T_s(x, t), T_i(x, t)) \cdots \) Temp. of snow, sea ice

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sea-ice

ocean
Thermodynamic Model by MU71

\[ F_a - \sigma(T_s(-h(t), t) + 273)^4 + \frac{k_s}{\partial x} \frac{\partial T_s}{\partial x}(-h(t), t) \]

\[ = \begin{cases} 
0, & \text{if } T_s(-h(t), t) < T_{m1}, \\
-q \dot{h}(t), & \text{if } T_s(-h(t), t) = T_{m1},
\end{cases} \]

\[ \rho S c_0 \frac{\partial T_s}{\partial t}(x, t) = k_s \frac{\partial T_s}{\partial x}^2(x, t), \quad \dot{h}(t) < x < 0, \]

\[ T_s(0, t) = T_i(0, t), \]

\[ k_s \frac{\partial T_s}{\partial x}(0, t) = k_0 \frac{\partial T_i}{\partial x}(0, t), \]

\[ (T_s(x, t), T_i(x, t)) \quad \text{Temp. of snow, sea ice} \]

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\( x \)
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\[ T_s(0, t) = T_i(0, t), \]
\[ k_s \frac{\partial T_s}{\partial x}(0, t) = k_0 \frac{\partial T_i}{\partial x}(0, t), \]
\[ \rho c_i(T_i, S) \frac{\partial T_i}{\partial t}(x, t) = k_i(T_i, S) \frac{\partial^2 T_i}{\partial x^2}(x, t) + I_0 \kappa_i e^{-\kappa_i x}, \quad 0 < x < H(t), \]

\((T_s(x, t), T_i(x, t)) \cdots \) Temp. of snow, sea ice

\(x\)

Air

Snow

Sea-Ice

Ocean

\(-h(t)\)

\(0\)

\(H(t)\)
Thermodynamic Model by MU71

\[ F_a - \sigma(T_s(-h(t), t) + 273)^4 + k_s \frac{\partial T_s}{\partial x}(-h(t), t) \]
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\[ T_s(0, t) = T_i(0, t), \]
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\[ \rho c_i(T_i, S) \frac{\partial T_i}{\partial t}(x, t) = k_i(T_i, S) \frac{\partial^2 T_i}{\partial x^2}(x, t) \]
\[ + I_0 \kappa_i e^{-\kappa_i x}, \quad 0 < x < H(t), \]
\[ T_i(H(t), t) = T_{m2}, \]
\[ qH(t) = k_i \frac{\partial T_i}{\partial x}(H(t), t) - F_w, \]
\[ (T_s(x, t), T_i(x, t)) \cdots \text{Temp. of snow, sea ice} \]
Thermodynamic Model by MU71

\[
F_a - \sigma (T_s(-h(t), t) + 273)^4 + k_s \frac{\partial T_s}{\partial x}(-h(t), t) = \begin{cases} 
0, & \text{if } T_s(-h(t), t) < T_m, \\
-q h(t), & \text{if } T_s(-h(t), t) = T_m,
\end{cases}
\]

\[
\rho_s c_0 \frac{\partial T_s}{\partial t}(x, t) = k_s \frac{\partial^2 T_s}{\partial x^2}(x, t), \quad -h(t) < x < 0,
\]

\[
T_s(0, t) = T_i(0, t),
\]

\[
k_s \frac{\partial T_s}{\partial x}(0, t) = k_0 \frac{\partial T_i}{\partial x}(0, t),
\]

\[
\rho c_i (T_i, S) \frac{\partial T_i}{\partial t}(x, t) = k_i (T_i, S) \frac{\partial^2 T_i}{\partial x^2}(x, t)
\]

\[
+ I_0 \kappa_i e^{-\kappa_i x}, \quad 0 < x < H(t),
\]

\[
T_i(H(t), t) = T_{m2},
\]

\[
q H(t) = k_i \frac{\partial T_i}{\partial x}(H(t), t) - F_w,
\]

**Salinity**

\[
S(x) = A \left[ 1 - \cos \left\{ \pi \left( \frac{x}{H(t)} \right)^{\frac{m+n}{H(t)}} \right\} \right]
\]

**Dependence**

\[
\begin{align*}
  c_i(T_i, S(x)) &= c_0 + \gamma \frac{S(x)}{T_i(x, t)^2}, \\
  k_i(T_i, S(x)) &= k_0 + \beta \frac{S(x)}{T_i(x, t)}
\end{align*}
\]

\[
(T_s(x, t), T_i(x, t)) \cdots \text{Temp. of snow, sea ice}
\]

\[
\begin{array}{c}
\text{air} \\
- h(t) \downarrow \\
0 \\
\text{snow} \\
H(t) \downarrow \text{x} \\
\text{sea-ice} \\
ocean
\end{array}
\]
Simulation Result by MU71

The diagram illustrates the seasonal variations in the thickness of snow, ice, and water layers over a period from January to December. The layers are labeled as Air, Snow, Ice, and Water. The isotherms are shown with different values indicating temperature variations. The diagram also includes time markers at the top for each month, and vertical scales for depth in centimeters.
Comparison with Empirical Data

Simulation (MU71)  Empirical Data (Untersteiner 1969)
Comparison with Empirical Data

Simulation (MU71)

Empirical Data
(Untersteiner 1969)
Problem Statement

- Problem
  1) Recent data shows no annual cycle.
  2) Complete profile of sea ice temperature is hard to measure.
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• Problem
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• Our Goal
  Estimate the temperature profile via available measurements.

• Method
  1) Design an estimator for simplified MU71 theoretically.
  2) Apply the estimator to original MU71 numerically.
State Estimation via Backstepping Observer

• Available Measurements

\[ \mathcal{Y}_1(t) = H(t), \]
\[ \mathcal{Y}_2(t) = T_i(0, t), \]
\[ \mathcal{Y}_3(t) = \frac{\partial T_i}{\partial x}(H(t), t). \]
State Estimation via Backstepping Observer

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\[ \mathcal{Y}_1(t) = H(t), \]
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- Simplified MU71

Salinity free: \( S(x) = 0 \)
State Estimation via Backstepping Observer

- **Observer Design**

\[
\hat{T}_i(0, t) = \mathcal{Y}_2(t),
\]

\[
\frac{\partial \hat{T}_i}{\partial t}(x, t) = D_i \frac{\partial^2 \hat{T}_i}{\partial x^2}(x, t) + I_0 \kappa_i e^{-\kappa_i x}
\]

\[
+ p(x, t) \left( \mathcal{Y}_3(t) - \frac{\partial \hat{T}_i}{\partial x}(\mathcal{Y}_1(t), t) \right), \quad 0 < x < \mathcal{Y}_1(t)
\]

\[
\hat{T}_i(\mathcal{Y}_1(t), t) = T_{m2}.
\]
State Estimation via Backstepping Observer

• Observer Design

\[ \hat{T}_i(0, t) = \mathcal{Y}_2(t), \]
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\[ \hat{T}_i(\mathcal{Y}_1(t), t) = T_{m2}. \]

• Error System

\[ \tilde{T}_i(0, t) = 0, \]
\[ \frac{\partial \tilde{T}_i}{\partial t}(x, t) = D_i \frac{\partial^2 \tilde{T}_i}{\partial x^2}(x, t) - p(x, t) \frac{\partial \tilde{T}_i}{\partial x}(H(t), t), \]
\[ \tilde{T}_i(H(t), t) = 0. \]
State Estimation via Backstepping Observer

- **Observer Design**

\[
\hat{T}_i(0, t) = \mathcal{Y}_2(t), \\
\frac{\partial \hat{T}_i}{\partial t}(x, t) = D_i \frac{\partial^2 \hat{T}_i}{\partial x^2}(x, t) + \bar{I}_0 \kappa_i e^{-\kappa_i x} \\
+ p(x, t) \left( \mathcal{Y}_3(t) - \frac{\partial \hat{T}_i}{\partial x}(\mathcal{Y}_1(t), t) \right), \quad 0 < x < \mathcal{Y}_1(t) \\
\hat{T}_i(\mathcal{Y}_1(t), t) = T_{m2}.
\]

- **Error System**

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\tilde{T}_i(0, t) = 0, \\
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\tilde{T}_i(H(t), t) = 0.
\]

Task: Derive \( p(x, t) \) to achieve \( \tilde{T} \to 0 \) quickly.
State Estimation via Backstepping Observer

- Backstepping Transformation

\[
\begin{align*}
  w(x, t) &= \tilde{T}_i(x, t) - \int_x^{H(t)} \nu(x, y)\tilde{T}_i(y, t)\,dy, \\
  \tilde{T}_i(x, t) &= w(x, t) - \int_x^{H(t)} n(x, y)w(y, t)\,dy,
\end{align*}
\]
State Estimation via Backstepping Observer

• Backstepping Transformation

\[ w(x, t) = \tilde{T}_i(x, t) - \int_x^{H(t)} \nu(x, y) \tilde{T}_i(y, t) dy, \]

\[ \tilde{T}_i(x, t) = w(x, t) - \int_x^{H(t)} n(x, y) w(y, t) dy, \]

• Target System

\[ w(0, t) = 0, \]

\[ \frac{\partial w}{\partial t}(x, t) = D_i \frac{\partial^2 w}{\partial x^2}(x, t) - \lambda w(x, t), \]

\[ w(H(t), t) = 0. \]
State Estimation via Backstepping Observer

- Backstepping Transformation
  \[ w(x, t) = \tilde{T}_i(x, t) - \int_x^{H(t)} \nu(x, y)\tilde{T}_i(y, t) dy, \]
  \[ \tilde{T}_i(x, t) = w(x, t) - \int_x^{H(t)} n(x, y)w(y, t) dy, \]

- Target System
  \[ w(0, t) = 0, \]
  \[ \frac{\partial w}{\partial t}(x, t) = D_i \frac{\partial^2 w}{\partial x^2}(x, t) - \lambda w(x, t), \]
  \[ w(H(t), t) = 0. \]

- Gain Derivation
  \[ p(x, t) = -\lambda x \frac{I_1 \left( \sqrt{\frac{\lambda}{D_i} \left( H(t)^2 - x^2 \right)} \right)}{\sqrt{\frac{\lambda}{D_i} \left( H(t)^2 - x^2 \right)}}, \]
  Online Calculation
Simulation Test of MU71
Simulation Test of MU71
Simulation of Temperature Estimation

- **Open-Loop Estimation**
- **Backstepping Observer**

![Graphs showing temperature estimation](image)

Faster Convergence
Gain Tuning

\[ \lambda = 0.5 \times 10^{-5} \]

\[ \lambda = 1.0 \times 10^{-5} \]

\[ \lambda = 5.0 \times 10^{-5} \]
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Gain tuning shows the tradeoff between convergence speed and overshoot.
Future Work

- Observer design with less measurements
- Comparison with a well-known estimator (e.g. Kalman filter)
- Implementation using empirical data.
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