Delay Compensated Control of the Stefan Problem

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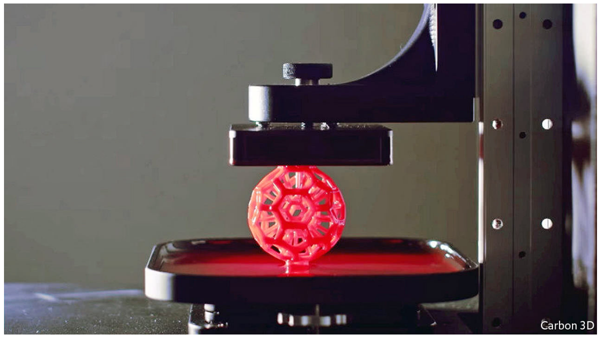
Motivation



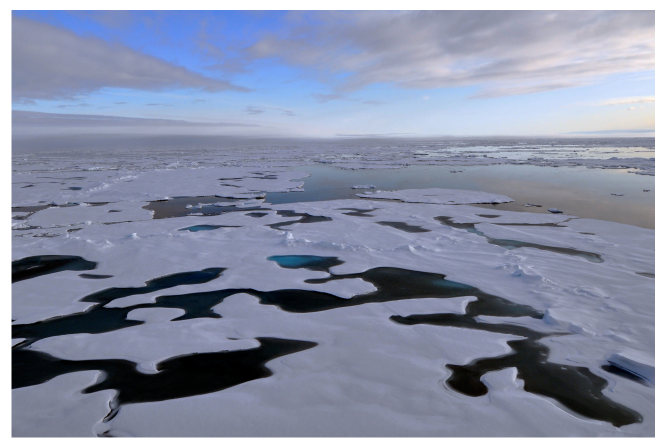
3D-Printing

Lithium Ion Batteries

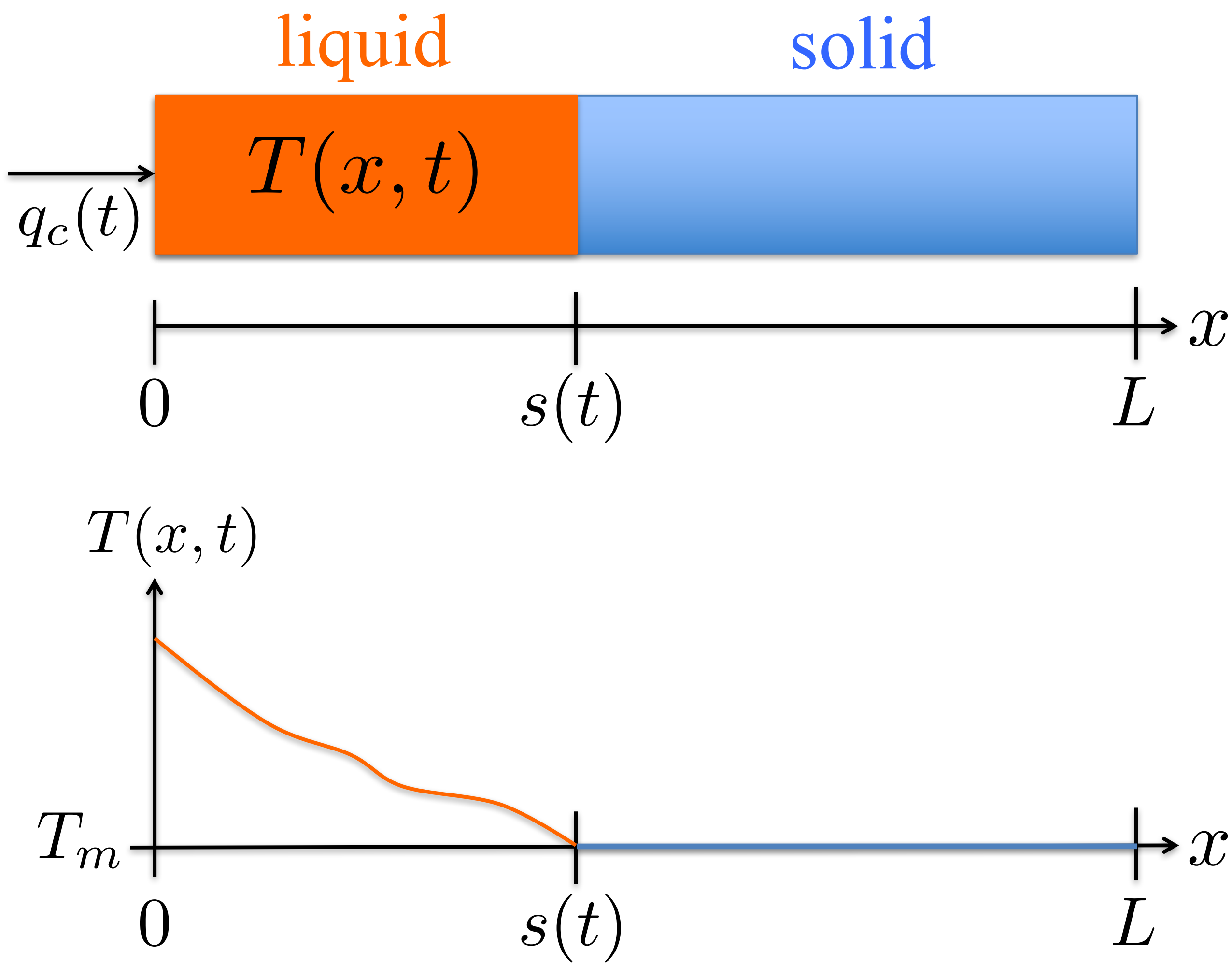
Stefan Problem (Phase Change Model)

Cryosurgery

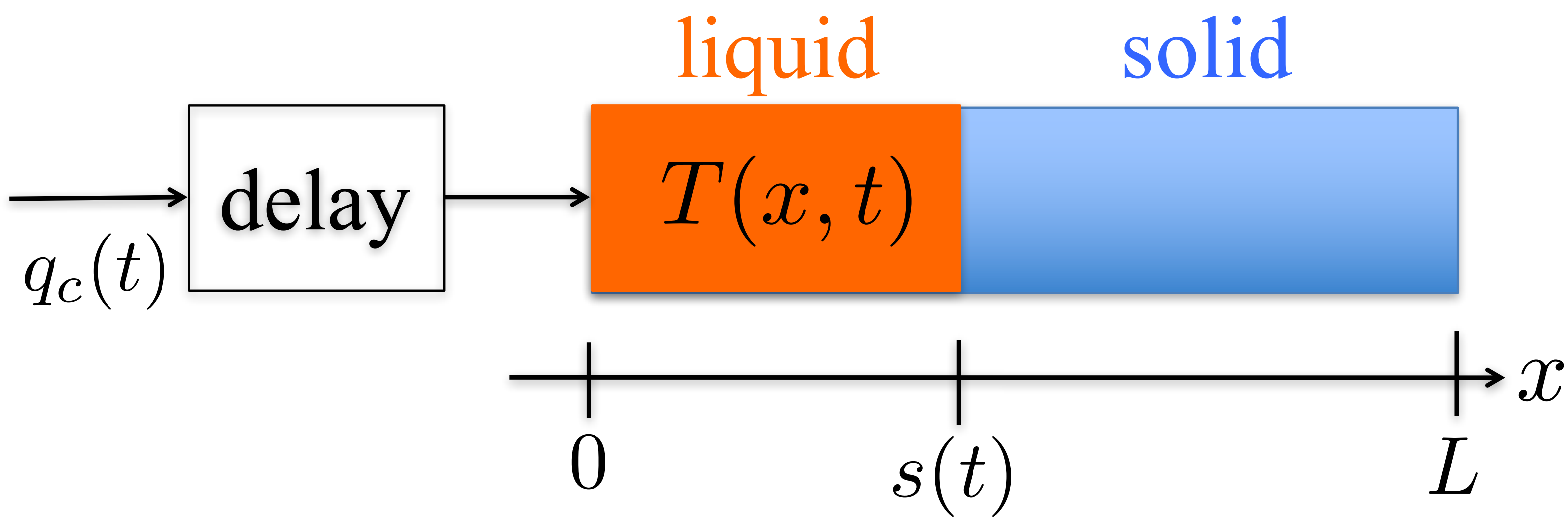
Sea Ice

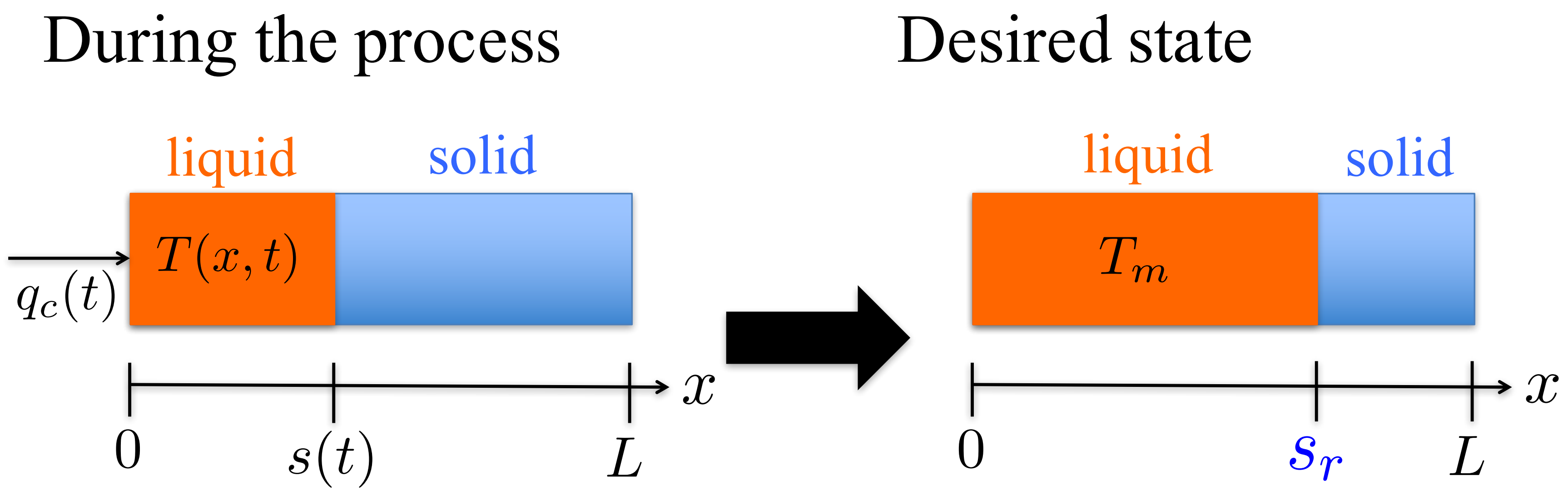


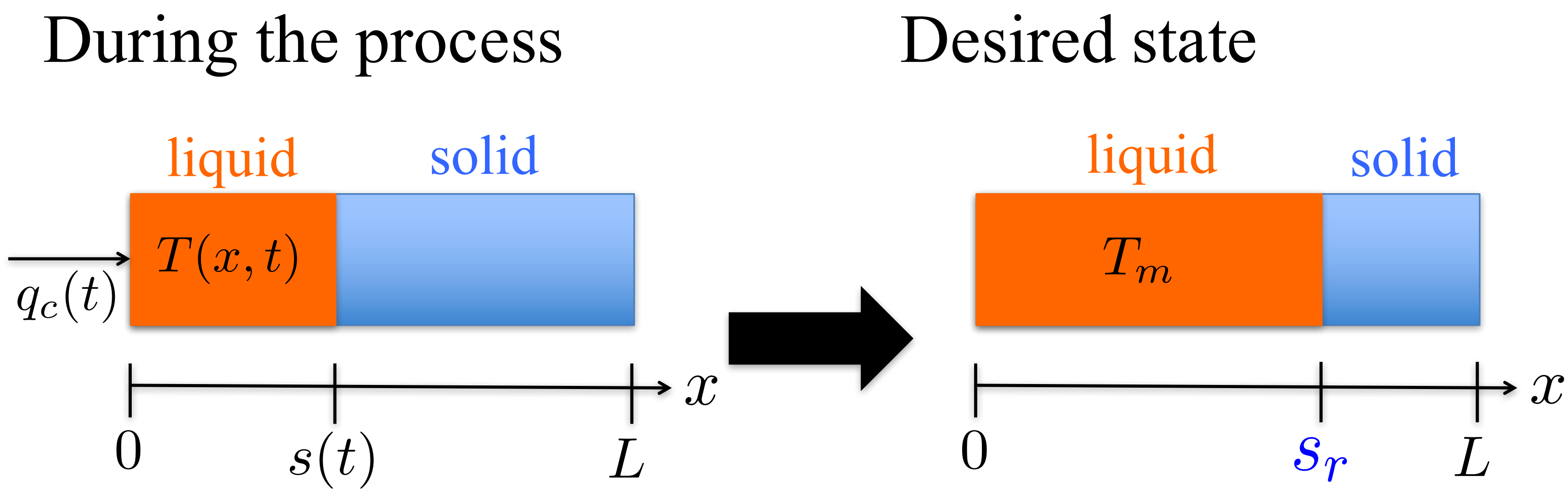
Physical Model : Melting



Physical Model : Melting + Actuator Delay

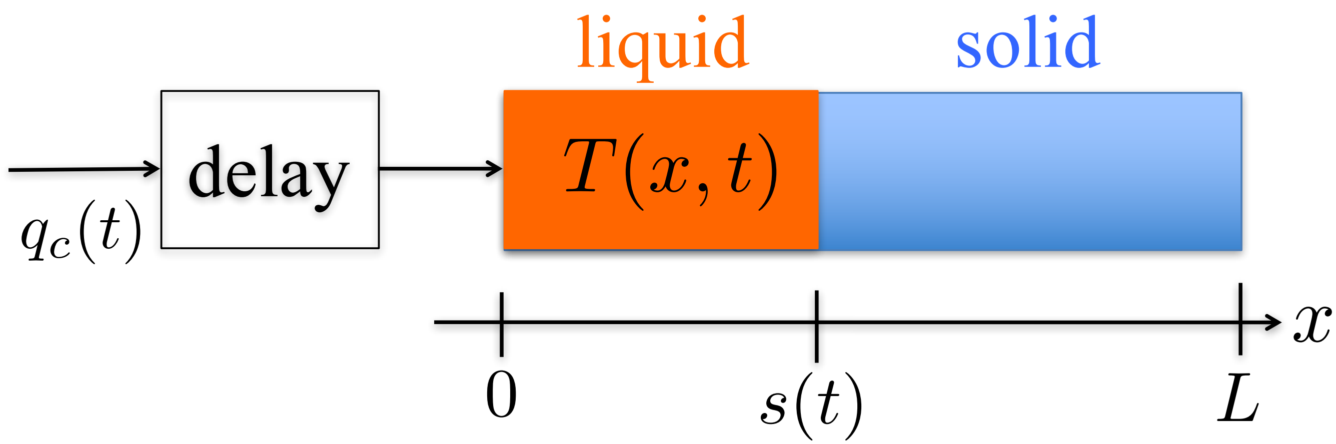



During the process
liquid
solid
$T(x,t)$
$q_c(t)$
0
$s(t)$
L
x
Desired state
liquid
solid
T_m
0
s_r
L
x

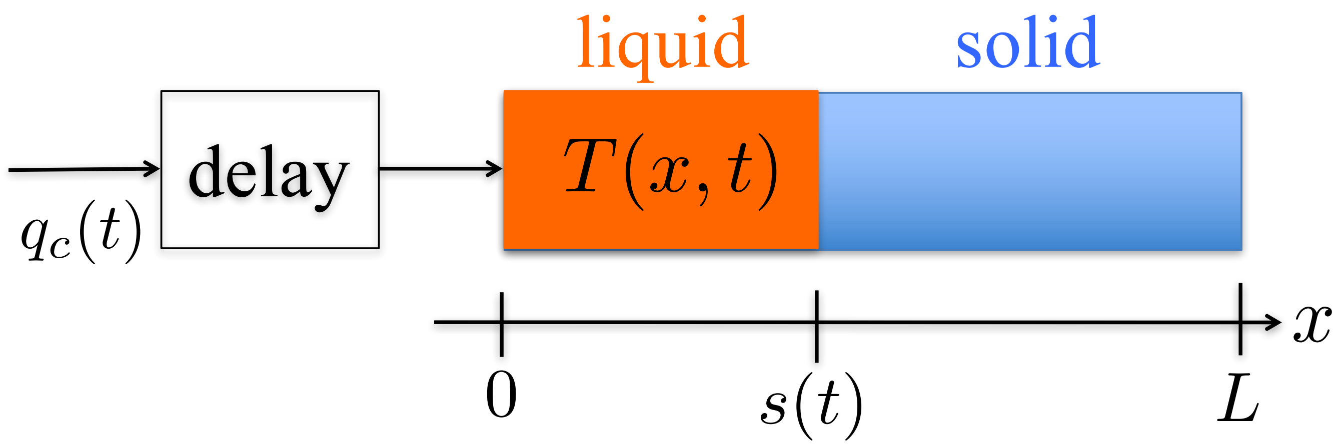


Objective: Design heat control $q_c(t)$ to achieve

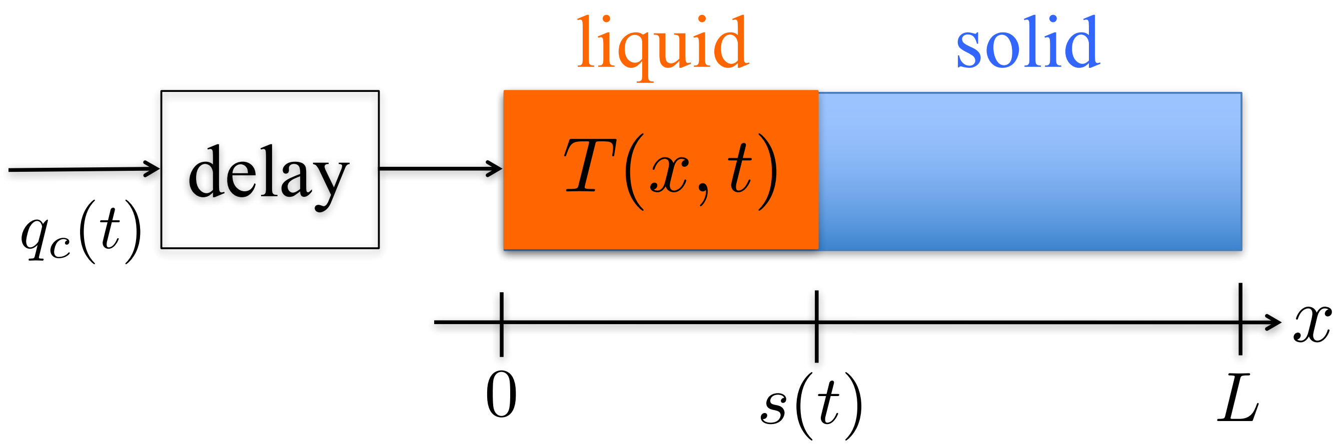
$$s(t) \to s_r, \quad T(x,t) \to T_m, \quad \text{as} \quad t \to \infty$$



liquid
solid
delay
$q_c(t)$
$T(x,t)$
x
0
$s(t)$
L



PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$



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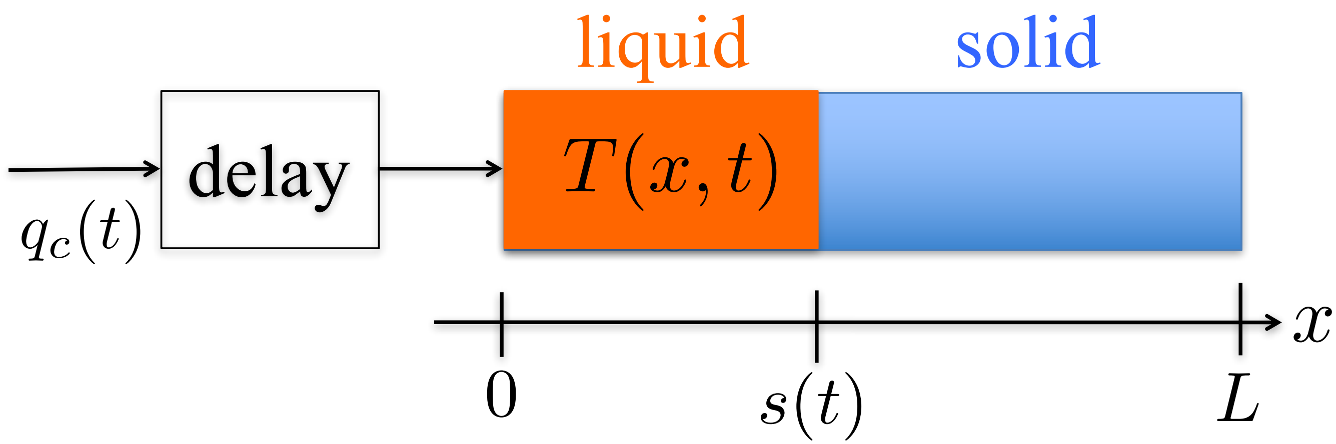
$$-kT_x(0,t) = q_c(t-D)$$

$$T(s(t),t) = T_m$$



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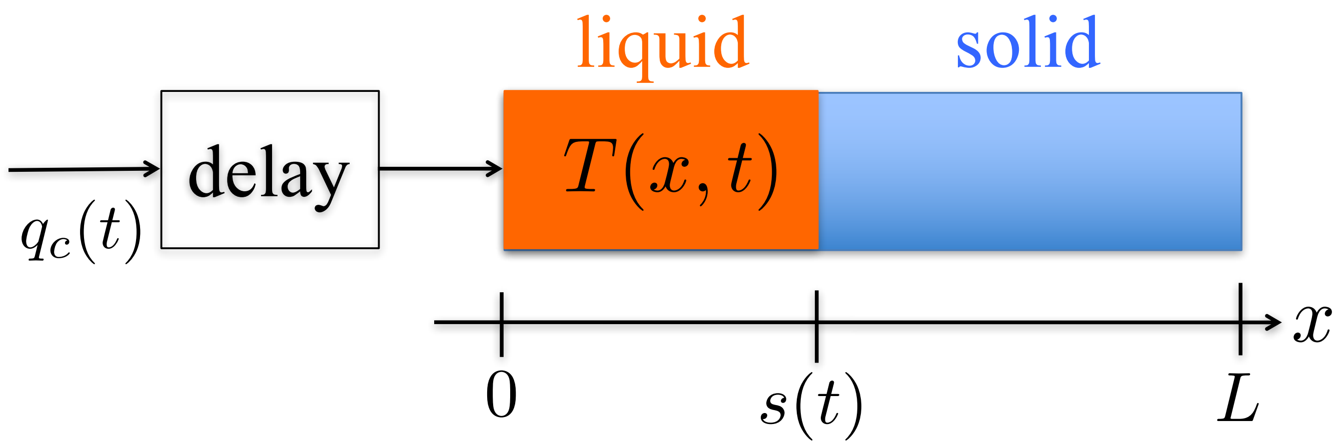
ODE $\quad \dot{s}(t) = -\beta T_x(s(t),t)$



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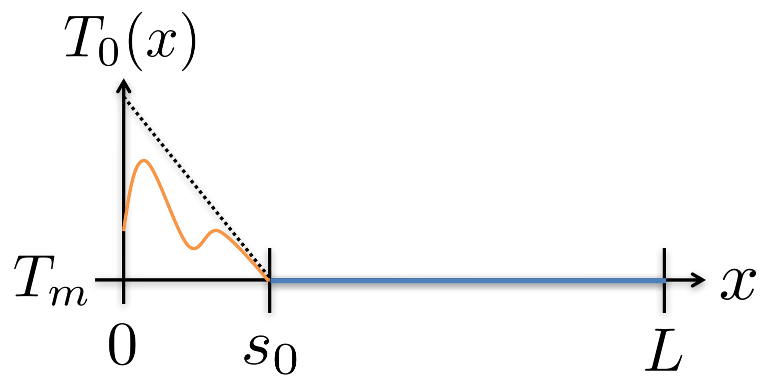
State-dependent moving boundary $\rightarrow$ Nonlinear

Assumption : Initial interface position $s_0 > 0$, and initial temperature $T_0(x)$ is Lipschitz (H := Lip. const.)

$$0 < T_0(x) - T_m < H(s_0 - x)$$



Assumption : The past input maintains non-negative, i.e.

$$q_c(t) \geq 0, \quad -D < \forall t < 0.$$

Model valid iff

$$T(x,t) > T_m, \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0$$

How to guarantee this?

Model valid iff

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How to guarantee this?

Lemma If $q_c(t) > 0$ $\forall t > 0$, then $\dot{s}(t) > 0$ $\forall t > 0$ and

$$T(x,t) > T_m, \forall x \in (0, s(t)), \forall t > 0$$

Energy Conservation

$$\frac{d}{dt}\left(\underbrace{\frac{k}{\alpha}\int_0^{s(t)}(T(x,t)-T_m)dx+\frac{k}{\beta}s(t)}_{\text{Internal Energy}}+\underbrace{\int_{t-D}^{t} q_c(\theta)d\theta}_{\text{Stored Energy}}\right)=\underbrace{q_c(t)}_{\text{Work}}>0$$

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- For model to be valid (single melting interface), heat must be added.

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- For model to be valid (single melting interface), heat must be added.
- When heat added, total energy (internal + stored) grows.
- Since total energy grows, energy corresponding to setpoint must be greater than initial energy

The following assumption **necessary** $\left(\text{because} \int_0^{s_r} (T_r(x) - T_m)dx = 0\right)$

Assumption : Setpoint s_r chosen to satisfy

$$s_r > s_0 + \beta \left(\frac{1}{\alpha} \int_0^{s_0} (T_0(x) - T_\mathrm{m})dx + \int_{-D}^{0} \frac{q_\mathrm{c}(t)}{k} dt \right)$$

Theorem The control law

$$q_{\mathrm{c}}(t) = -c\left(\int_{t-D}^{t} q_{\mathrm{c}}(\theta)d\theta + \frac{k}{\alpha}\int_{0}^{s(t)}(T(x,t)-T_{\mathrm{m}})dx + \frac{k}{\beta}(s(t)-s_{\mathrm{r}})\right),$$

where $c > 0$, makes the closed-loop system globally exponentially stable in the norm

$$||T - T_m||^2_{\mathcal{H}_1} + (s - s_r)^2.$$

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Note : Control law is nonlinear because of $s(t)$ in integration limit.

Explanation of Design

Reference errors

$$u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r$$

Change of variable

$$v(x,t) := q_{\mathsf{C}}(t - x - D)/k$$

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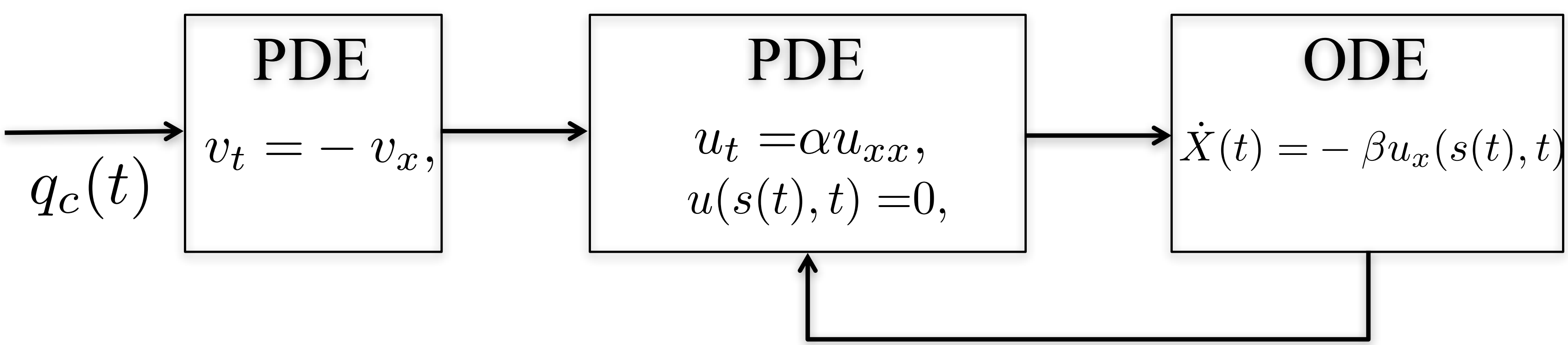
Change of variable

$$v(x,t) := q_c(t - x - D)/k$$

(v, u, X)-system

$$\begin{aligned} v_t(x,t) =& -v_x(x,t), \quad -D < x < 0 \\ v(-D,t) =& q_c(t), \\ u_x(0,t) =& -v(0,t), \\ u_t(x,t) =& \alpha u_{xx}(x,t), \quad 0 < x < s(t) \\ u(s(t),t) =& 0, \\ \dot{X}(t) =& -\beta u_x(s(t),t). \end{aligned}$$



Explanation of Design

Backstepping transformations

$$w(x,t) = u(x,t) - \frac{c}{\alpha}\int_x^{s(t)} (x-y)u(y,t)dy - \frac{c}{\beta}(x-s(t))X(t)$$

$$z(x,t) = v(x,t) + c\int_x^0 v(y,t)dy + \frac{c}{\alpha}\int_0^{s(t)} u(y,t)dy + \frac{c}{\beta}X(t)$$

$\rightarrow$ both are *nonlinear* transformations

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Target system

$$z_t(x,t) = -z_x(x,t), \quad -D < x < 0$$
$$z(-D,t) = 0,$$
$$w_x(0,t) = -z(0,t),$$
$$w_t(x,t) = \alpha w_{xx}(x,t) + \frac{c}{\beta}\dot{s}(t)X(t), \quad 0 < x < s(t)$$
$$w(s(t),t) = 0,$$
$$\dot{X}(t) = -cX(t) - \beta w_x(s(t),t).$$

Model validity

Proposition Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$.

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Proof:

$$q_c = -c\,(\text{total energy} - \text{setpoint energy})$$

$$\dot{q}_c(t) = -c q_c(t) \quad \therefore q_c(t) = q_c(0) e^{-ct} > 0$$

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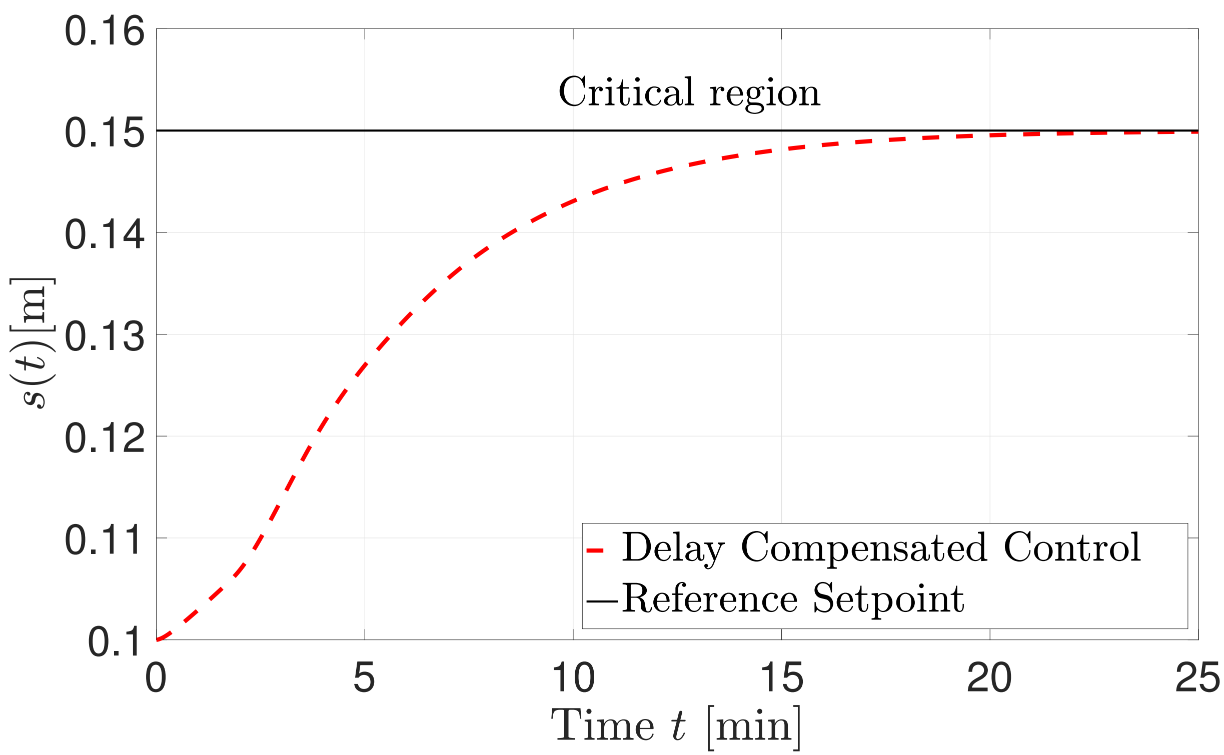
Lyapunov analysis with $\dot{s}(t) > 0$ and $s_0 < s(t) < s_r$ yields the norm estimate

$$\Psi(t) \leq M\Psi(0)e^{-bt},$$

for some positive constants $M > 0$ and $b > 0$,
where $\Psi(t) = ||v||^2_{\mathcal{H}_1(-D,0)} + ||u||^2_{\mathcal{H}_1(0,s(t))} + X(t)^2$.

Numerical Simulation

Zinc

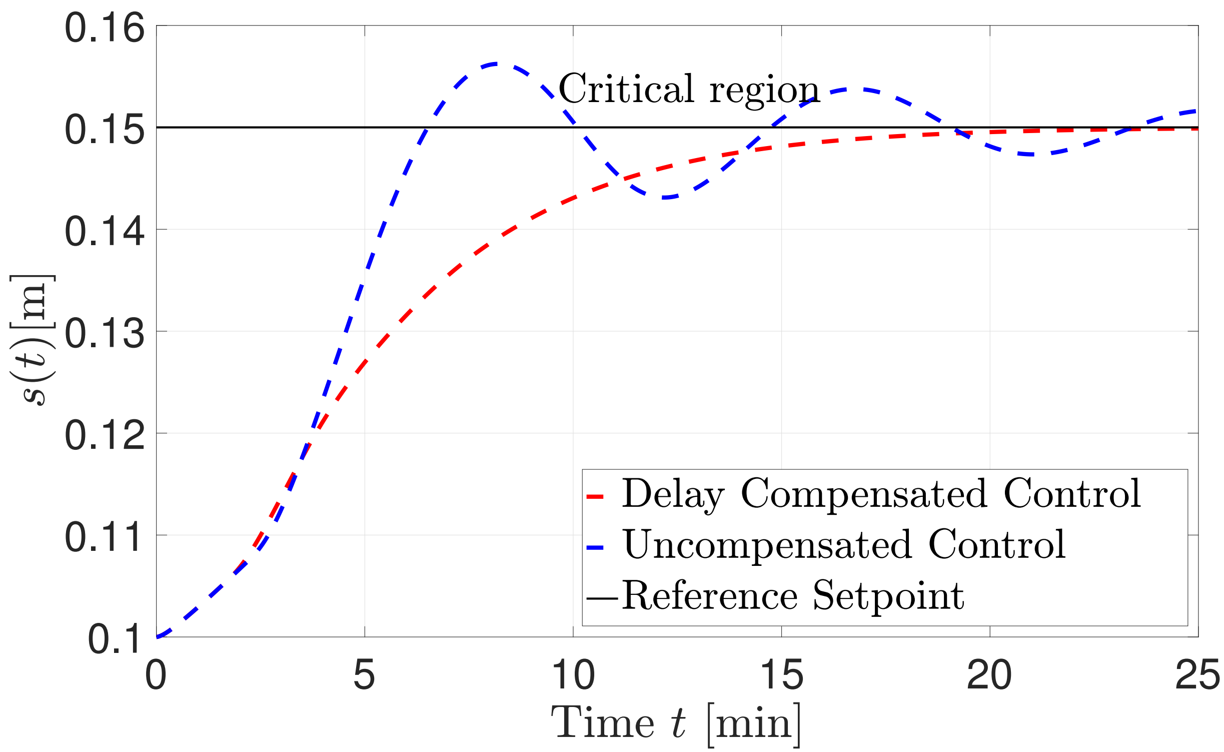


No overshoot

Numerical Simulation

Compare with uncompensated control

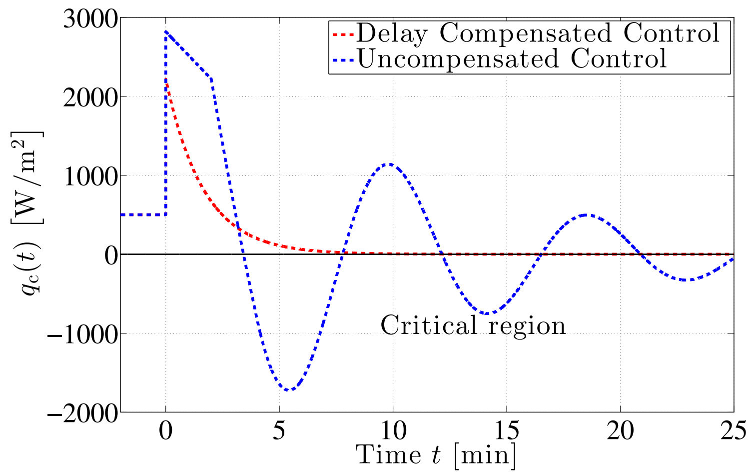
$$q_{\mathrm{c}}(t) = -c\left(\frac{k}{\alpha}\int_0^{s(t)}(T(x,t)-T_{\mathrm{m}})dx + \frac{k}{\beta}(s(t)-s_{\mathrm{r}})\right),$$



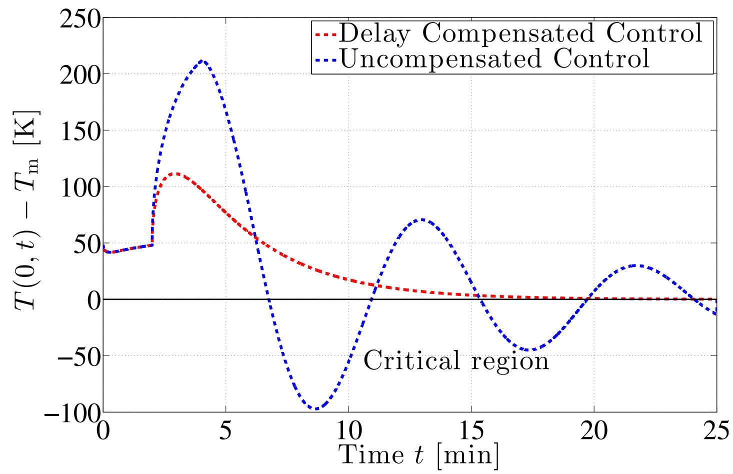
Uncompensated control violates the model validity

Numerical Simulation

Heat input



Temp. at x=0



Uncompensated control violates the model validity

Future Work

- Observer design under sensor delay

- Adaptive control under unknown delay