Control of 2-Phase Stefan Problem via Single Boundary Heat Input

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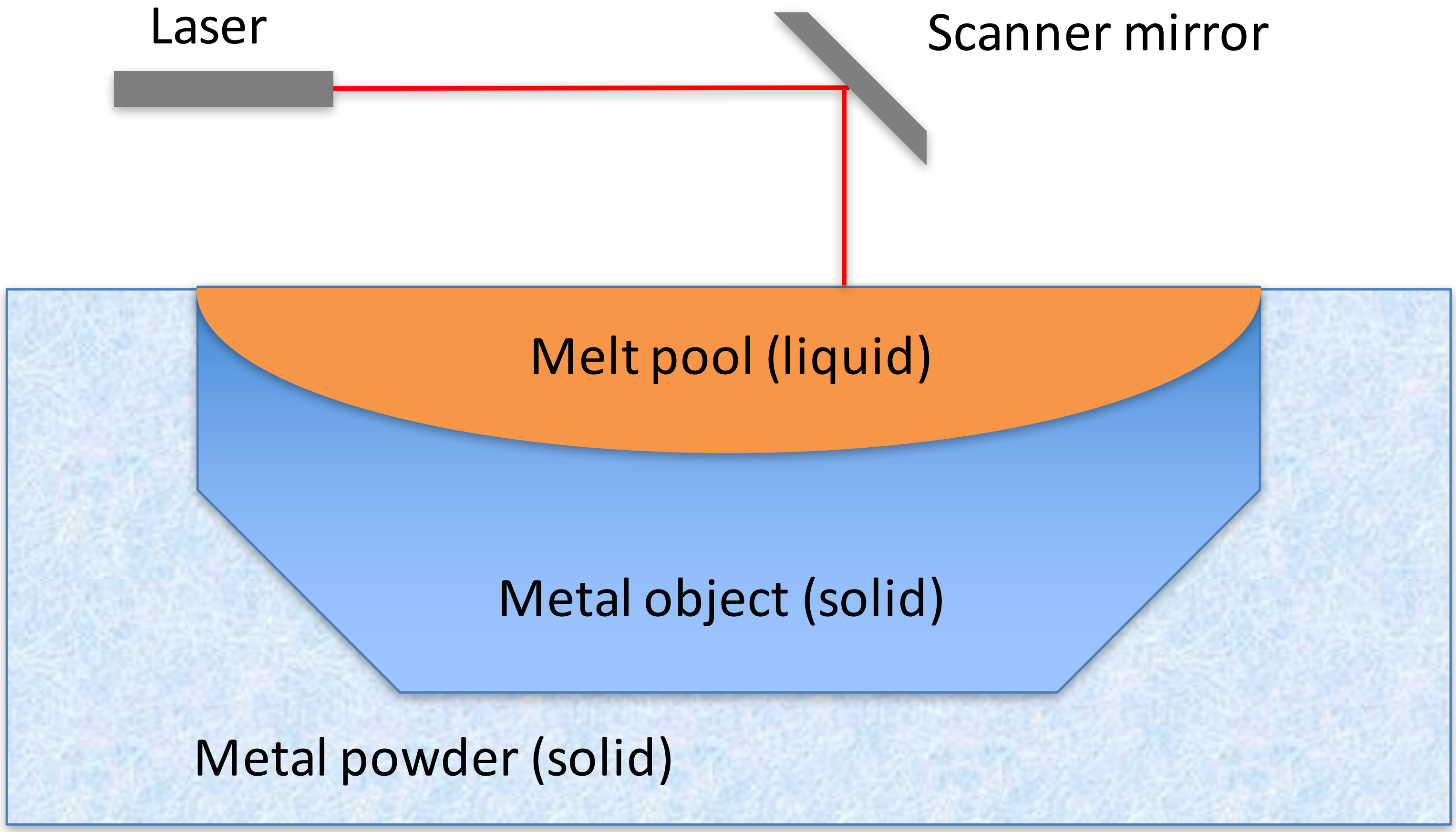
Motivation

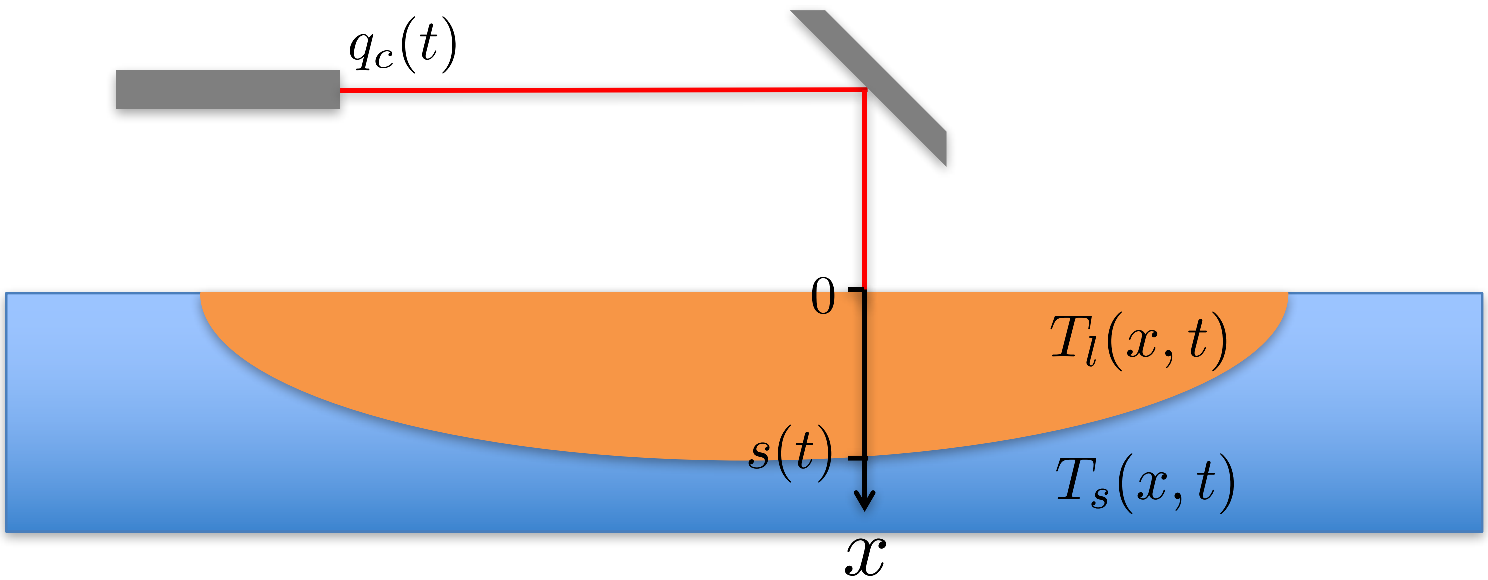
Metal 3D-printing with selective laser sintering



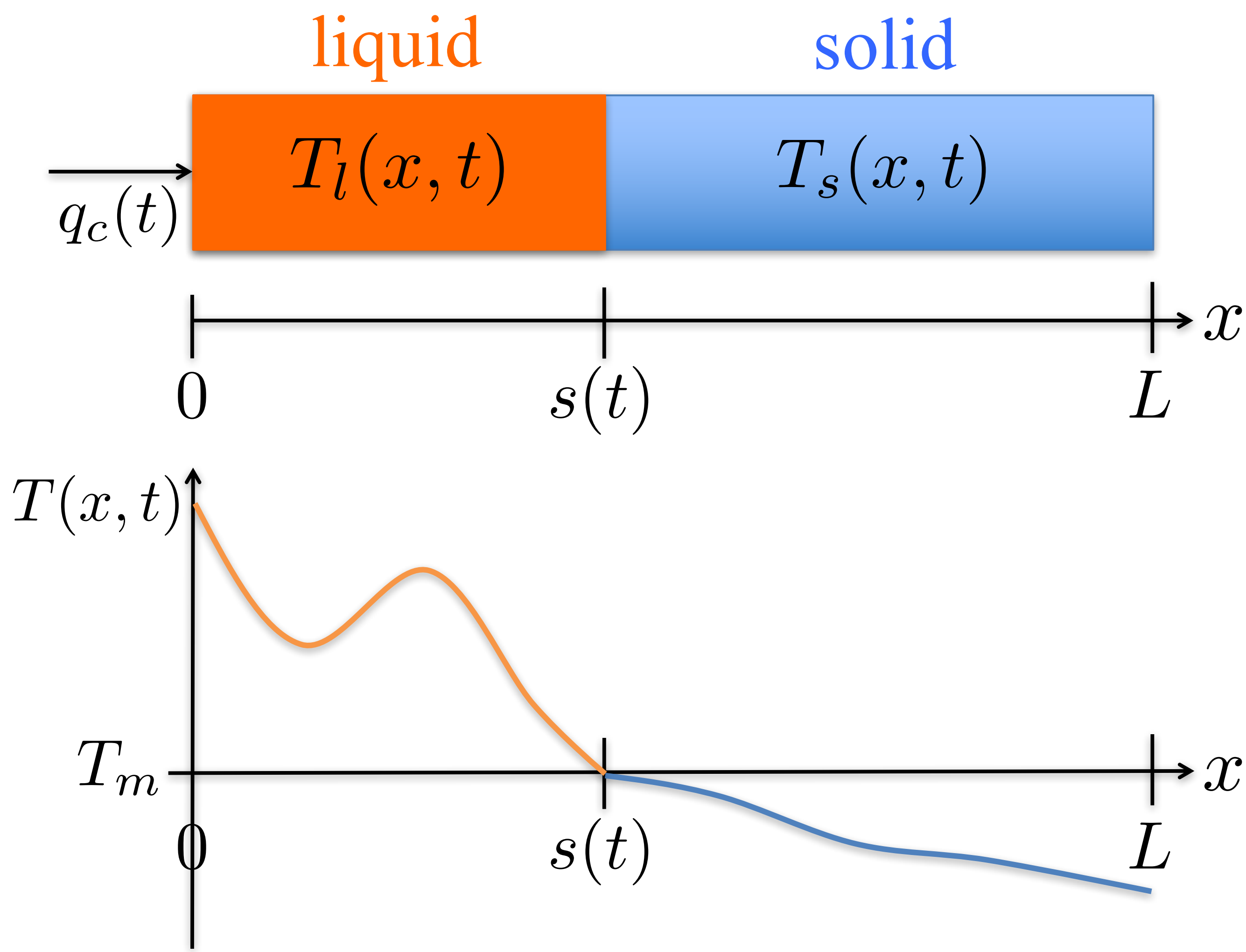
⋆ Repetitive process of melting and solidification

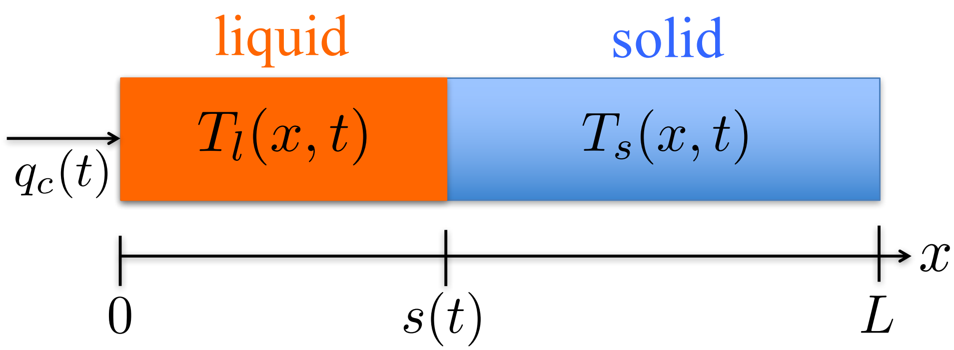
Physical model

⋆ [Chung & Das, 2004] developed physical model by two-phase Stefan problem



Schematic of temperature dynamics with phase change

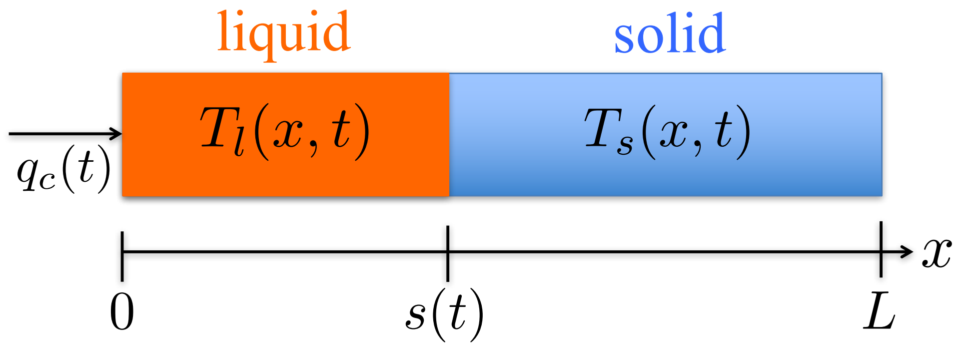




Liquid PDE

$$\frac{\partial T_l}{\partial t}(x,t) = \alpha_l \frac{\partial^2 T_l}{\partial x^2}(x,t), \quad 0 < x < s(t),$$

$$\frac{\partial T_l}{\partial x}(0,t) = -q_c(t)/k_l, \quad T_l(s(t),t) = T_m,$$



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$$\frac{\partial T_s}{\partial t}(x,t) = \alpha_s \frac{\partial^2 T_s}{\partial x^2}(x,t), \quad s(t) < x < L$$
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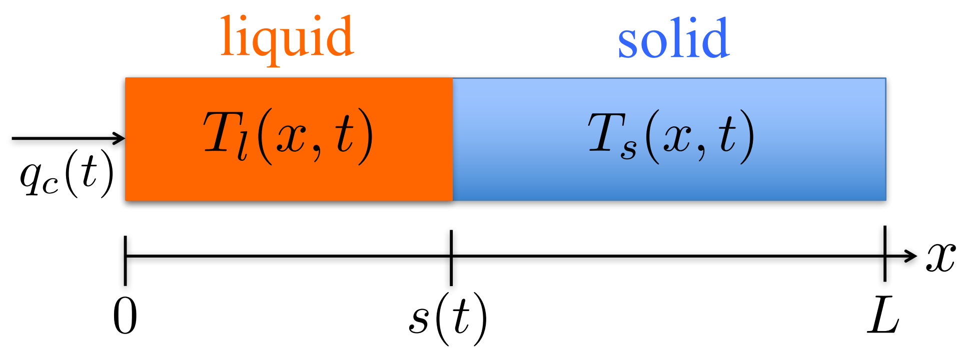

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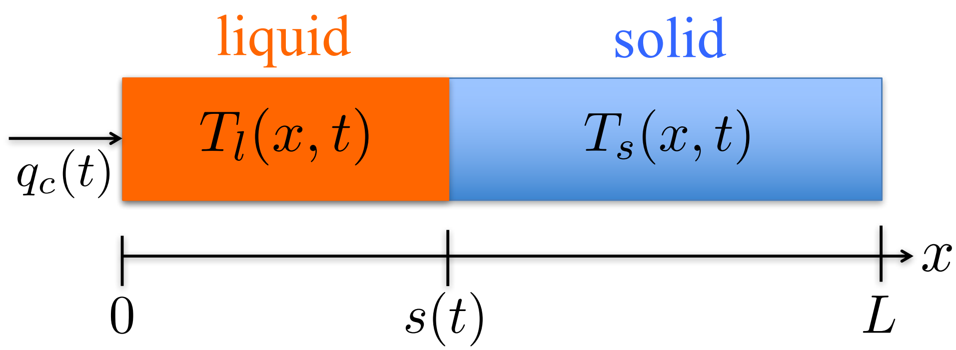

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Goal : Design $q_c(t)$ to drive $s(t) \to s_r$

Remark Model valid iff

$$
\begin{aligned}
T_l(x,t) \geq& T_m, \quad \forall x \in (0, s(t)), \quad \forall t > 0, \\
T_s(x,t) \leq& T_m, \quad \forall x \in (s(t), L), \quad \forall t > 0, \\
0 <& s(t) < L, \quad \forall t > 0.
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Lemma If $q_c(t) > 0$, then above conditions are satisfied *except for* $s(t) < L$

Assumption The setpoint is chosen to satisfy

$$\underline{s}_r < s_r < L$$

where $\underline{s}_r := s_0 + \frac{\beta_l}{\alpha_l} \int_0^{s_0} (T_{l,0}(x) - T_m)dx + \frac{\beta_s}{\alpha_s} \int_{s_0}^{L} (T_{s,0}(x) - T_m)dx$
and $\beta_i = \frac{k_i}{\gamma}$ for $i = l, s$.

⋆ $\underline{s}_r$ is the final interface position with $q_c(t) \equiv 0$.

Energy Conservation Law

Internal energy (specific heat $+$ latent heat)

$$E(t) = \frac{k_l}{\alpha_l}\int_0^{s(t)} (T_l(x,t) - T_m)dx + \frac{k_s}{\alpha_s}\int_{s(t)}^{L} (T_s(x,t) - T_m)dx + \gamma s(t)$$

satisfies (energy growth = external work)

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Energy Shaping (ES) Control

$$q_c(t) = -c(E(t) - E_r)$$

drives $E(t)$ to a setpoint energy $E_r = \gamma s_r$ with satisfying $q_c(t) > 0$ & $s(t) < L$.

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Design by ES, and prove stability by *backstepping*

Theorem The control law (ES)

$$q_c(t) = -c\left(\frac{k_l}{\alpha_l}\int_0^{s(t)}(T_l(x,t)-T_m)dx \right.$$

$$\left. +\frac{k_s}{\alpha_s}\int_{s(t)}^{L}(T_s(x,t)-T_m)dx + \gamma(s(t)-s_r)\right),$$

where $c > 0$, makes the closed-loop system globally exponentially stable in

$$(s-s_r)^2 + ||T_l - T_m||_{L_2}^2 + ||T_s - T_m||_{L_2}^2$$

Sketch of Proof:

Reference Error PDE Variables

$$u(x,t) := T_l(x,t) - T_m, \quad v(x,t) := T_s(x,t) - T_m,$$

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Error System

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two PDEs and ODE $\rightarrow$ complicated.......

Absorb solid PDE into ODE

$$X(t) := s(t) - s_r + \frac{\beta_s}{\alpha_s} \int_{s(t)}^{L} v(x,t)dx.$$

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$\star$ Prove stability of (u, X) at $(0,0)$ under ES design of $q_c(t)$.

State Transformation (from (u, X) to (w, X))

$$w(x,t) = u(x,t) - \frac{\beta_l}{\alpha_l}\int_x^{s(t)} \phi(x-y)u(y,t)dy - \phi(x-s(t))X(t),$$

where $\phi(x) = \frac{1}{\beta_l}(cx - \varepsilon)$.

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Target (w, X)-System

$$\begin{aligned} w_t(x,t) =& \alpha_l w_{xx}(x,t) + \frac{c}{\beta_l}\dot{s}(t)X(t), \\ w(s(t),t) =& \frac{\varepsilon}{\beta_l}X(t), \\ \dot{X}(t) =& -cX(t) - \beta_l w_x(s(t),t). \end{aligned}$$

and, by ES design,

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which can be rewritten by (w, X) on RHS using inverse transformation.

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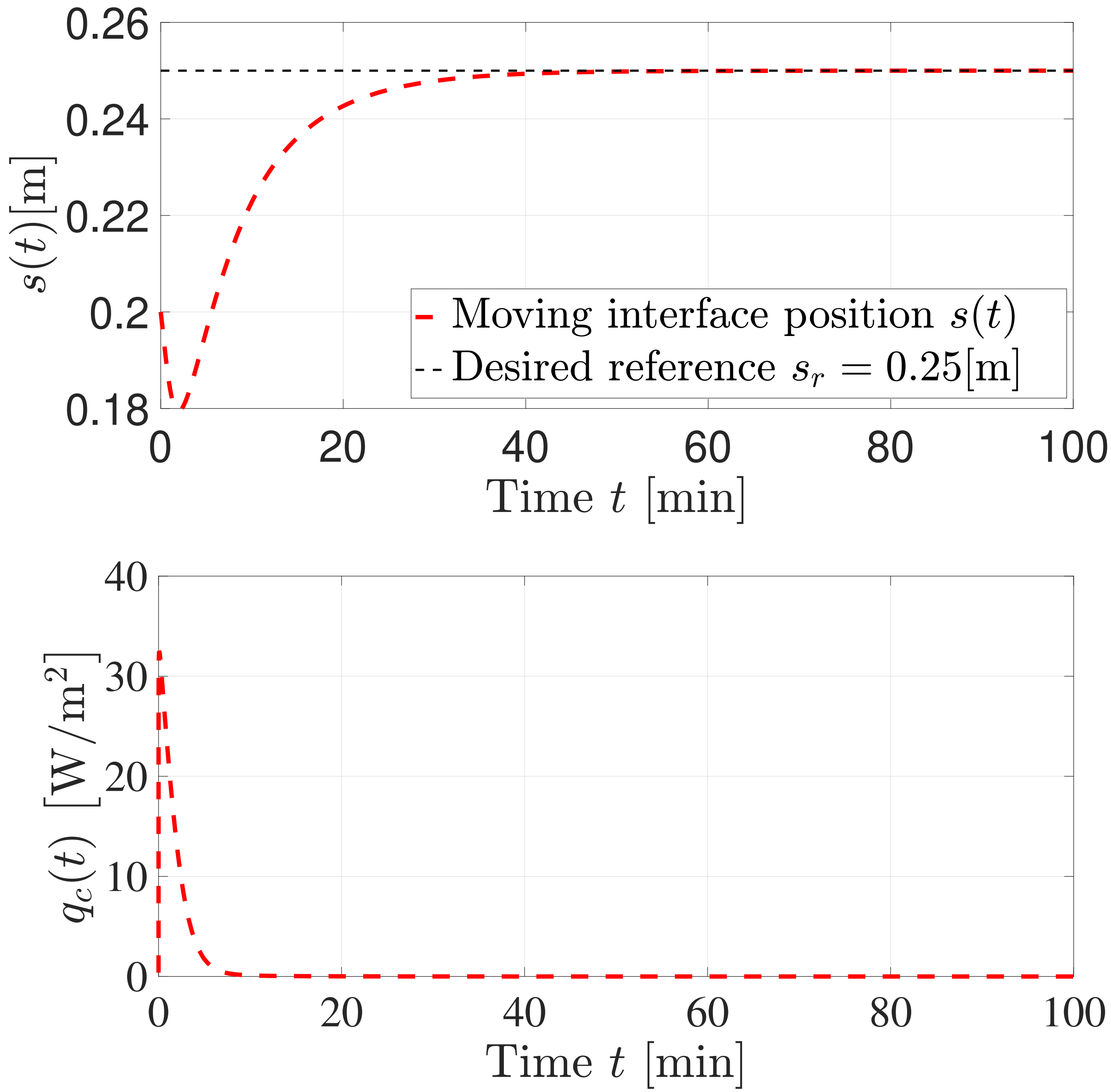
Prove g.e.s. of (w, X)-system by Lyapunov method $\rightarrow$ concludes Theorem

Numerical Simulation with Zinc



Future Direction

- Application to metal AM by incorporating the penetration of laser (Beer's law)

$$\frac{\partial T_l}{\partial t}(x,t) = \alpha_l \frac{\partial^2 T_l}{\partial x^2}(x,t) + e^{-ax} q_c(t),$$

- Observer to estimate temperature profiles and interface position (challenging)